Elements of Financial Economnics

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1 Introduction

This course deals with the pricing of ønancial assets when economic agents are rational and markets are organized in a **competitive** fashion. Here such agents are allocate part of their real economic resources into ønancial assets (such as stocks, bonds etc.) in order to obtain the most preferred pattern of consumption between the present and the future. This results in certain norms which should hold on the aggregate in a community (i.e. for a large number of individuals) and which can be used to obtain the □fair□ prices of ønancial assets. Fair prices reÆect the tastes of all the members of the community (where individual peculiarities have washed out), their individual endowments in resources, and the relative scarcity of resources in the aggregate.

To be sure the□fair□ prices might not be the ones that will be necessarily observed in real markets. They only yield a measure of how far actual prices reÆect the salient features of the economy. In addition, they provide a benchmark for pricing of new ønancial assets, i.e. assets which have not been already □tested□ by the ønancial markets.

2 Decision Making Under Certainty

The economic environment considered here is the following: To make the problem simple we assume that there are two time-periods (present and future) and one commodity (say, corn). In each period the society produces, consumes and invests in corn[1]. An individual lives for only two periods and has a given **initial endowment** of resources, i.e. corn, (e_o, e_1) for each of the two periods. Note, that corn available in the present and corn available in the future are two distinct commodities.

For the moment ignore production and suppose that the society receives its initial endowments like manna from heaven. If an individual wishes to obtain more corn for the future than the quantity he/she is endowed with, he/she can buy it in today□s market in exchange for present corn. In addition, the individual can obtain future corn by lending present corn to some other member of the community. The loan involves a certain **interest rate** (r), and it will be repaid in the future in terms of corn. Because here there is no uncertainty, buying (or, selling) corn in today□s market should be equivalent to borrowing (or, lending). The two activities should yield the

[1]Alternatively, the society deals in baskets of real world commodities,.such as baskets as they appear in GDP statistics.

same result, and the market price of future corn (p) must have a special relation to the rate of interest. In fact, if one speciøes that the price of corn in today□s market is equal to one, arbitrage between borrowers-buyers and lenders-sellers will guarantee that

$$p = \frac{1}{1+r},$$

where r is the interest rate on loans. Equivalently, r is the rate of return of a **risk-free bond** denoted in terms of corn.

The concern of the individual economic agent is to choose how much corn to consume now and how much in the future. The decision depends on his/her **preferences for present vs. future consumption** and the **constraints** imposed by the initial endowents in corn.

To make the problem concrete, some further assumptions are required. First, we specify that the market for corn is organized in a **competitive** fashion; i.e. no individual buyer or seller of corn can inÆuence its prices. Next, individuals always prefer more corn than less, and their preferences display decreasing **marginal rate of substitution** (MRS). This guarantees that the indi□erence curves between present and future corn are downward slopping and concave upwards [2].

$$\cancel{y = e_1 + e_0(1+r) - (1+r)x}$$

To start with, note that he individuals□present wealth if he/she consumes nothing in the furutre is:

$$W_0 = e_0 + \frac{e_1}{1+r}$$: the max amount of Present wealth (ie wealth if he consumes nothing in the future)

and the future wealth if he/she consumes nothing in the present is:

$$W_1 = e_1 + e_0(1+r)$$: max future wealth if he consumes nothing in the present.

The straight line joining W_0 and W_1 in Figure 1 below is called the **transformation frontier** and shows the combinations of present and future wealth which is made possible through borrowing and lending. The equation of the line is

$$\begin{aligned} x_1 &= W_1 - (1+r)x_0 \\ &= (W_0 - x_0)(1+r) \end{aligned}$$

where x_1 stands for future wealth and x_0 for present wealth. The slope of the transformation frontier is constant and equal to $-(1+r)$.

[2] MRS is the rate at which the individual is willing to substitute on the margin present corn for future corn and it is measured as the ratio of marginal utility of future corn to that of persent corn.

Mathematically, the problem of the agent can be expressed as the choice of present and future consumption (c_0, c_1), so as to maximize the consumption preferences of the individual subject to the constraints imposed by the initial endowments. In formal terms, the agent is assumed to solve the following **maximization problem**:

$$\text{maximize } u(c_0, c_1)$$

$$\begin{aligned} \text{subject to } c_0 + pc_1 &= e_0 + \frac{1}{1+r} e_1 \\ &= W_0 \end{aligned}$$

where u is a two-period **utility function,** representing the consumption preferences of the agent .

Deøne λ to be the **Lagrangean multiplier**. The Lagrangean expression for this problem is:

$$L = u(c_0, c_1) + \lambda[e_0 + pe_1 - c_0 - pc_1],$$

and the ørst order conditions:

$$\begin{aligned} u_{co} &= \lambda \\ u_{c1} &= \lambda p \end{aligned}$$

where u_{co} and u_{c1} are the **partial derivatives** of u with respect to c_0 and c_1, respectively.

By taking ratios of the above expressions we have:

$$-\frac{\partial c_0}{\partial c_1} = \frac{u_{c1}}{u_{co}} = \frac{1}{1+r} = p. \tag{1}$$

The solution of the ørst order conditions together with the budget constraint determine the optimal consumption bundle (c_0^*, c_1^*) : At the optimum the MRS between present and future consumption (which is the slope of the indiﬀerence curve) is equal to the price of future corn.

As explained above, the MRS signiøes the rate at which the agent is willing to substitute on the margin present consumption for future consumption. On the other hand, p is the rate at which one can trade present quantities of the commodity in exchange for future ones. Then, the above expression states that on the margin the agent can do through trading what he is willing to do.

Graphically, the optimal consumption can be determined at the point where an **indifference curve** is tangent to the transformation frontier, as shown in Figure 1.

Figure 1:

2.1 Real Investment Opportunities

Suppose that the agent has an initial endowment of only present corn $(e_0, 0)$. Future corn can be obtained through production and through borrowing or lending, as before. The problem now is to devise a plan that maximizes the agents□consumption preferences with the added possibility of producing corn.

One way to deal with this problem is to solve it in two steps:

∘Step 1: First, the agent decides what is optimal to produce given the possilibities open through borrowing and lending and, then

∘Step 2:what to consume.

Suppose that in the ørst step the agent uses an amount $I \equiv (e_0 - x_0)$ as seed (i.e. **investment**) in order to produce future corn x_1. The part that is not invested (x_0) and x_1 can be traded in today□s market in the same manner as the initial endowments of the previous section . In fact it helps to think of (x_0, x_1) as (modiøed) initial endowments. Then, the solution of the second step is exactly as it was explained in the previous section where there was no production. The ability to solve the problem in two steps has been called the□**Fisher Separation Theorem**□ and depends crucially on the absence of transaction costs and the assumption that borrowers and lenders face the same interest rate[3].

Solution of the ørst step: Assume that in production takes place under **constant returns to scale** [4]. Let

$$F(x_0, x_1) = 0 \tag{2}$$

represent the **maximum** amount of future output (x_1) that can be obtained by investing $I \equiv (e_0 - x_0)$ of corn in the ørst period. The graph of this relation in

[3]This assumption is is not unrealistic under the present conditions. It is equivalent to the "law of one price" which requires that any commodity commands one price in the market.

[4]Constant returns make the structure of production irrelevant: Large corporations are equivalent to the existence of many small production units.

called the **production possibility frontier (PPF)**. The construction of the PPF assumes that the best available technology is used, so that corn is not wasted, and that there are no gifts of corn from outside the system. It slops downward to indicate that if we want more present corn we must be contented with less future corn. It is **concave** to the origin to indicate that the more future corn we wish to produce, the more present corn we must sacriøce on the margin. This property is called the **diminishing marginal rate of transformation (MRT)**.

Mathematically, to determine the optimum (x_0, x_1) the agent:

$$\text{maximizes } x_0 + \frac{x_1}{1+r}$$

$$\text{subject to } \ F(x_0, x_1).$$

Note that the objective function is the present wealth of the agent if we interpret (x_0, x_1) as the modiøed initial endowments.

The **Lagrangean** of this problem is:

$$L = x_0 + \frac{x_1}{1+r} - \lambda[F(x_0, x_1)]. \tag{3}$$

ørst order conditions of the optimization problem are:

$$\begin{aligned} 1 &= \lambda F_{x_0} \\ \frac{1}{1+r} &= \lambda F_{x_1} \end{aligned}$$

where F_{x_0} and F_{x_1} are the partial derivatives of F with respect to x_0 and x_1 and their ratio gives the MRT between present and future corn. By dividing these two conditions we obtain:

$$\frac{1}{1+r} = \frac{F_{x_0}}{F_{x_1}} = p$$

which states that in equilibrium the MPT is equal to the price of corn. In other words one can obtain the same amount of future corn by sacriøcing one unit of present corn whether one engages in production or in borrowing-lending. That is, the agent is indi□erent between these two activities.

Solution of the second step: The solution is identical to the one of the previous section if we interpret (x_0, x_1) as the new initial endowments. As before, it states that the MRS in consumption is equal to the future price of corn. Combining both results we obtain:

$$\frac{u_{c_1}}{u_{c_0}} = \frac{F_{x_0}}{F_{x_1}} = \frac{1}{1+r} = p$$

i.e. the MRS is equal to the price of future corn and to the the marginal rate of transformation (MRT) of corn in production: In equilibrium the agent can transform present corn into future corn at the rate he/she is willing to do, and it is irrelevant whether he/she so by engaging in production or trading in the market.
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$$\text{maximizes } u(x_0, x_1)$$

$$\text{subject to } F(x_0, x_1).$$

The **Lagrangean** of this problem is:

$$L = u(x_0, x_1) - \lambda[F(x_0, x_1)]. \tag{4}$$

ørst order conditions of the optimization problem are:

$$\begin{aligned} u_{co} &= \lambda F_{x_0} \\ u_{c1} &= \lambda F_{x_1} \end{aligned}$$

where F_{x_0} and F_{x_1} are the partial derivatives of F with respect to x_0 and x_1..The ratio of these conditions state that the MRS of corn in consumption is equal to the marginal rate of transformation (MRT) of corn in production:

$$\frac{u_{c_1}}{u_{c_0}} = \frac{F_{x0}}{F_{x_1}}.$$

Again, the agent is able to substitute, on the margin, present consumption for future consumption at the rate he is willing to by ørst choosing the right point on the production possibility frontier, and then moving where his/her preferences are maximized on the transformation frontier. For example in Figure 2 the optimal plan of the individual is the following: Invest $(e_0 - x_0)$ to produce x_1 output of future corn . This leaves x_0 initial corn to be disposed in the present. Optimum present consumption is c_0^*. Thus, $x_0 - c_0^*$ can be lended-out. In the future the individual will consume c_1^* which will come from the outpt of his/her investment (x_1), plus $(e_0 - x_0)(1 + r) = c_1^* - x_1$ from the loan (that is from the principal plus interest of the amount lended in the past).

To have as much corn as possible to trade in the market, the individual must reach the furthest out point on the transformation frontier that can be reached. This is acieved by producing up to the point where his/her production possibility frontier has the same slope as the transformation frontier: $-(1 + r)$. In other words, given the slope of the TF, the individual selects a point on the PPF having the greatest

Figure 2:

present value. This determines what to produce. Then, using the (modiøed) initial endwments he/she borrows or lends to achieve the optimum (c_o^*, c_1^*).

In a market economy the interest rate indicates the rate at which the community wishes to substitute present consumption for future one. The fact that the interest rate is common to everyone, means that in **equilibrium** the MRS of all community members are equal to $(1+r)$,[5]. In other words the individual selects a point on the

Additional topics to be discussed:

·Beneøts of Capital Markets

·Transaction Costs: The Fisher Separation Theory does not apply when there are transaction costs; the outcome depends on preferences.

·Firm: Investment decisions and the □unanimity□ of shareholders]]]]

3 Uncertainty-Risk

Chance events, or random events: Events that occur unsystematically, without a speciøc pattern, plan, or connection.

Random Experiment: A proceedure (or, a series of observations) which lead to an outcome that cannot be predicted.

Random mechanism: The supposed force operating during an exprtiment or observation that makes certain phenomena (or, outcomes) happen in a particular way without any apparent cause. A mechanism that produces a sequence of events which, each one taken separately, cannot be predicted. The collection of all possible outcomes (or, phenomena) that may materialize during a random exprtiment or

[5] Note, having the same MRS does not mean that all members consume the same bunddle (c_o, c_1). It means that in equilibrium their indifference curves are tangent to a transformation frontier with slope $\frac{1}{1+r}$.

observation forms the **Sample space** of the experiment. We will call the primitive outcomes of an experiment **elementary events** (or points) of the sample space. They may be ønite or inønite in number. Here we will be concerned exclusively with spaces having ønite, or countably inønite points. The sample space and its elementary events are particular to the experiment performed. A subset of the sample space is called an **event**.

Example: an outcome that depends on the throw of one coin relates to a sample space with two elementary events, H,T (H for heads, and T for tails). The outcome depends on the throw of two coins relates to a sample space with elementary points HH, HT, TH, TT.

Probability of an elementary event: A number associated with an elementary event. It indicates the likelyhood of that event to occur. Their properties are: (a) the probability of an event is a number bewteen (and including) zero and one, (b) the sum of probabilities of all events in the sample space is equal to one.

Example: In an experiment involving the throw of a dice, elementary events are the six faces that may turn up. If the dice is not biaced (i.e. it does not tend to favour one face over the other) the probability associated with each face is equal to 1/6.

Probalility equal to one: indicates certainty (i.e a degenerate sample space). Probability equal to zero: indicates impossibility.

More general events are deøned as the union of points in (or, subsets of) the sample space. More generally, they are deøned as the union of other events. Their probability is the sum of the probabilities of the elementary events included in that event.

An event in the example of the throw of one dice is the following:

□The face that will show up has value less than three□. This consists of the elementary events that one or two will show up; i.e. it is the union of those two elementary events, and its probability is 2/6 (i.e 1/6+ 1/6).

(1x)

Mutually Exclusive Events ~~Mutually Independent events: Events that~~ do not have any elementary events in common, i.e. the intersection of their unions is empty. The probability of both events occuring is the sum of their probabilities. When the events are not independent, the probabitity of both events occuring is the sum of their probabilities minus the probability of both occuring at the same time; i.e. the probability of the elementary events they have in common. ok

Statistically Independent experiments (observations, trials etc.): Experiments in which the realization or not of an event is not a□ected by the fact that the same event was observed or not previously. Example: the outcome that depends on the throw of a dice twice in a row. The fact that eg. three came up in the ørst throw does not hinder or facilite the appearence of three in the second throw.

define conditional probability

A random variable ($\tilde{x}(\xi)$) is a function that associates some numerical value (ξ) to every outcome of an experiment. Here the domain of the function is the set elementary events and their unions(i.e. the set of all subsets of the sample space) and

the range is the numerical values ξ. (In the notation of a random variable the range is usually omitted.) Example the throw of a dice with pay-o□s: In a gabbling game the face of the dice that shows up is associated with a pay-o□: eg. if one shows up you loose \$1, if two you loose \$2, and if three you lose \$3. If four shows up you win \$1, if øve you win \$2, and if six you win \$3. It follows that the event □face two□ is accociated with -2. The event □either face two or three□ is associated with 1,

Consider a random variable $\widetilde{x}$ and some number x, $-\infty < x < \infty$. The cumulative distribution function of $(\widetilde{x}(\xi)$ is a function giving the probability that the number associated with the random variable $\widetilde{x}$ is less than or equal to x for every value of x.

Formally,

$$F(x) = \Pr(\widetilde{x}(\xi) \leqslant x \quad \text{for} \quad -\infty < x < \infty. \tag{5}$$

Loosely speaking, we consider the set of all subsets of the sample space that are associated with a numerical value $\xi \leqslant x$. The cumulative distribution function gives the sum of the probabilities of those subsets. ~~Of course, the union of those subsets~~ is ~~an event~~.

4 Elements of the Expected Utility Theory

Life demands that people make choices and each choice has certain consequences or, outcomes. The choice from among several alternatives is based on the preferences of the individual over the **outcome** of the choice. In some situations, the outcome is clearly deøned (such as a pare vs. an apple). In others, it is to unpredictable to a certain degree ~~; e.g. an apple under certain circumstances or a pare under others~~.

Explain

To make these concepts more precise we deøne the following:

A **state of the world** is a complete description of the economic environment. Any future environment is described by a collection of states each one having some probability to occur; the collection of states is exhaustive. In addition, any two states are mutually exclusive, i.e.they are distinct. Suppose that there are a fi finite number (n) of such states. For the purposes of this course, we will assume that each state results in an outcome which is measured in units of the consumption commodity; e.g., units of corn as in the previous section. Consider a □contract□ that speciøes the number of consumption units its owner will receive if a particular state occurs and denote it with the vector $x = (x_1, x_2, ..., x_n)$. This contract together with the vector denoting the probability of each state to occur , $p = (p_1, p_2, p_3, ..., p_n)$, is called **a prospect**. Another name for prospects is **lottery or probability distribution**. A prospect is called **certain,** (or **risk-free**), if it yields the same outcome in each state of the world; i.e. a risk-free prospect is a degenerate lottery.

As saide above choices depend on the preferences of the individual for the outcome, but preferences is a neboulus concept. It is desirable thet they can be quantiøed ine following sense: Consider the choice between two objects A and B that belong to a

given set, and suppose that A is perferred A to B. One would like to attach a number (or **utility index**) to the intensity of this preference, so that the more prefered is the object the higher is the number associated with it. Here, the number (or utility index) of A should be greater than that of B.

Choice under uncertainty refers to situations where the outcome of the choice is to a certain degree unpredictable[6]. In this course we describe uncertainty in terms of states of the world and prospects, as explained above. The **Theory of Expected Utility** is the dominant theory that quantiøes preferences for choice under uncertainty: It assumes that when the individual is faced with a choice from among a number of prospects, is he/she is able to **compare** any two of them and decide which one he/she prefers. Preferences over prospects are also assumed to be **consistent**; i.e. in making choices the individual is not expected to contradict the ones made previously. These two basic assumptions, together with a few axioms developed be von-Neumann and Morgestern (N-M), form the backbone of the theory. These axioms impose some rather technical requirements, with the exception of the **Axiom of Independence** which gives the theory its empirical content.

The basic idea behind the axiom of independence is as follows: Consider three prospects A, B and C, and let prospect A be preferred to B. Construct two **compound** lotteries (i.e. lotteries whose outcomes are lotteries themselves): one which yields (A,C) with probability $(p, 1-p)$, and one which yields (B,C) with the same probabilities. Then, the individual should prefer the ørst lottery over the second. This is so because (judged separately) A is preferred to B, and C is has the same probability to occur in both lotteries. This seems reasonable enough, but it has been found to be frequently violated in experimental applications of the theory.

The implications of the N-M axioms can be explained as follows, albeit with great violation to the theory. Consider prospects deøned over a set of possible outcomes X; eg. one such prospect A yields outcome $x = (x_1, x_2, ..., x_n)$ from the set X, with probability $p = (p_1, p_2, p_3, ..., p_n)$). First, the theory asserts we can ønd a function (u), called the **von-Neumann- Morgestern utility function (N-M)**, which assigns to A the index:

$$E[u(x_A)] = \sum_i p_i u(x_i).$$

Eg. for $x = (x_1, x_2)$ and $p = (p, 1-p)$ the expected utility of A is

$$E[u(x_A)] = pu(x_1) + (1-p)u(x_2).$$

If there are inønite states and outcomes, the expected utility index becomes:

$$E[u(x_A)] \equiv \int_X u(x)f(x)dx = \int_X u(x)dF(x)dx.$$

[6]The terms "uncertainty" and "risk" refer to different situations and they will be clarified later. For the time being they will be used interchanably.

where $f(x)$ is the density function, and F is the cumulative distribution function of the prospect. It is seen that the utility index assigned to A is linear in probabilities. This is a consequence of the Axiom of Independence and it gives the theory its empirical content.

Second, the indices assigned represent the preferences of the individual over these prospects in the following sense: for any two prospects A and B, if A is at least as desirable as B, then:

$$E[u(x_A)] \geq E[u(x_B)].$$

A further implication of the linearity of the expected utility index in the probabilites is the following: If $E[u(x_A)]$ is a N-M utility index, so is $E[V(x_A)] \equiv a + bE[u(x_A)]$ for any two constants a and b. In other words, any linear transformation of an expected utility function is also an expected utility function. To verify this statement follow the calcuations of the expected utility index of prospect A. Substitute $pu(x_1) + (1-p)u(x_2)$ for $E[u(x_A)]$ and add & subtract ap

$(1-p)$

$$\begin{aligned} E[V(x_A)] &\equiv a + bE[u(x_A)] \\ &= a + b[pu(x_1) + (1-p)u(x_2)] \quad (6) \\ &= p[a + bpu(x_1)] + [a + b(1-p)u(x_2)] \quad (7) \\ &= pV(x_1) + (1-p)V(x_2). \quad (8) \end{aligned}$$

From this it can be easily veriøed that if $u(x)$ represents the preferences of an individual so does $V(x)$; i.e.

$$E[u(x_A)] \geq E[u(x_B)] \text{ implies } E[V(x_A)] \geq E[V(x_B)]. \quad (9)$$

Just a matter of notation: We usually denote a prospect by the vectors x and p, as said above. We will also use the notation $x(F)$ to denote the dependence of a prospect on the cumulative distribution of its outcomes.

A useful diagram to visualize prospects (i.e. probability distributions) in the case there are only three outcomes can be constructed as follows: Consider the prospect (x_1, x_2, x_3), (p_1, p_2, p_3). Hold (x_1, x_2, x_3) constant and let (p_1, p_2, p_3) change[7]. Note that probabilities add-up to one and, because the states are exhaustive, $p_2 = 1 - p_1 - p_3$.Use the triangle as in the diagram below to locate di□erent prospects, i.e. di□erent probabilities for the given returns (x_1, x_2, x_3). In Figure 3, there are two prospects shown: one (x) which yields (x_1, x_2, x_3) with probabily $p_1 = 1/5, p_2 = 2/5$ $(= 3/5 - 1/5)$, $p_3 = 2/5$, and another (y) with the same outcomes as the ørst one but with probabilities $p_1^{'} = 2/5$, $p_2^{'} = 1/5$, $p_3^{'} = 2/5$.

We can draw indi□erence curves among the prospects (i.e. the probability distributions) represented in the diagram for a given utility function. The expression for an indi□erence curve becomes:

$$p_1 u(x_1) + (1 - p_1 - p_3)u(x_2) + p_3 u(x_3) = k$$

[7]That is the set of possible outcomes X contains only one point $x = (x_1, x_2, x_3)$.

Figure 3:

Figure 4:

for some constant k. Solving for p_3 we obtain:

$$p_3 = \frac{k - u(x_2)}{u(x_3) - u(x_2)} - \frac{u(x_1) - u(x_2)}{u(x_3) - u(x_2)} p_1.$$

Thus, the indi□erence curves are linear, and if $u(x_3) \succ u(x_2) \succ u(x_1)$ they are slopping upwards. Their slope is:

$$\frac{dp_3}{dp_1} = \frac{u(x_2) - u(x_1)}{u(x_3) - u(x_2)},$$

where $u's$ are constant. Thus changing the utility level k, only shifts the curves in a parallel fashion.

It can be shown that the more concave the utility function is the greater the slope.

4.1 The Utility Function

The functional form of the utility function $u(x)$ is of great signiøcance in representing the attitude to risk of the individual. First it is assumed that its ørst derivative,

$u'(x)$ is non-negative; i.e. the inividual always prefers more of x than less. Second, the sign of the second derivative signiøes whether the individual is willing to pay to avoid risk, is una□ected by the presence of risk, or seeking to assume risk. In the ørst case utility is a concave function, its second derivative is negative and the individual is said to be a **risk averter.** In the second the utility is a linear function, its derivative is zero and the individual is **risk neutral**. In the third case it is a convex function, its derivative is positive and the individeual is **risk seeker** (or, **risk loving**). Before we proceed, it is useful to deøne some concepts.

Deønition. The **certainty equivalent** $CE(x)$ of a prospect A is a number which satisøes the following:

$$u(CE(x)) = E[u(x_A)].$$

✓

i.e. the individual is indi□erent between the prospect A and an amount $CE(x)$ with certainty.

Deønition. The **risk premium** of a prospect is:

$$q(x_A) \equiv \mu(x) - CE(x))$$

✓

where $\mu(x) \equiv \int_c xf(x)dx$ is the mathematical expectation of the returns of the prospect.

The following diagram shows graphically the expected utility of a prospect A that yields $x = (x_0 = 5, x_1 = 30)$ with probability $(p = 2/5, 1 - p = 3/5)$. If the graph of the utility function has the shape assumed in the diagram, the expected utility of x is:

$$E[u(x_A)] = \frac{2}{5}u(5) + \frac{3}{5}u(30).$$

Note that the mean (or, **expected return**) of the prospect is \$20, and its utility is $u(20)$.

To give a concrete example, suppose the utility function of Figure 5 is the **negative exponential** function $(1 - e^{-0.1*x})$. Its second derivative is: $-0.1^2e^{-0.1*x} < 0$, therefore it is a concave function. With this function $u(5) = 0.393$, $u(30) = 0.950$, ✗

✓ and $E[u(x)] =$ .7275 (Note that $E[u(x)]$ is 3/5□s the diastance between $u(30)$ and $u(5)$). The expectation of the prospect is 20units of the consumption good and

✓ $u(20) =$ 0.8647 Since $u(20) > E[u(x)]$ the ørst conclusion is that the individual prefers the expected return rather than the prospect itself. By how much? Looking at the diagramme, we see that there is a risk-free amount that has the same utility index as $E[u(x_A)]$. This amount is the certainty equivalent of x $(CE(x))$, and in this case $x =$ 13.001 The second conclusion is that the individual is indi□erent between

Figure 5:

13.01

the prospect A and ~~11.874~~ units of the consumption good; and, thus, prefers the expected return of the prospect (i.e. 20 units) to the prospect itself. This is a general result for all concave utility functions. The di□erence between the expected return and the certainty equivalent of x (i.e. $20 - 13.001 = 6.999$) is the risk premiuim. Note XY
that te risk premium is positive. This is so because, loosely speaking, the graph of the function of Figure 3 lies above the straight line that joins $u(5)$ and $u(30)$.

Similar reasoning will show that when the function is convex (linear) the individual prefers (is indi□erent to) the prospect to (and) its expected return. This is so because when $u(x)$ is convex (linear) the graph of the function in Figure 3 lies below (coinsides with) the straight line that joins $u(5)$ and $u(30)$, and the risk premium is negative (zero).

We can summarize the above discussion with the following statement:

?

Conclusion 1 ~~Conjecture 2 Conclusion~~n 3 A necessary and su□ cient condition for an individual to be X

- o risk-adverse is that $u''(x) < 0$, or equinalently, $q(x) > 0$,
- o risk-neutral is that $u''(x) > 0$, or equinalently, $q(x) = 0$,
- o risk-loving is that $u''(x) = 0$, or equinalently, $q(x) < 0$.

No

4.1.1 Stochastic Dominance

First Degree Deønition. Given two cumulative distribution functions F and G, we say F dominates G in the ørst degree (**FSD**) if $F(x) \leq G(x)$, for all x which belong to a given set of outcomes C.

This property is shown in the following diagrams for the discrete and the continuous distribution, respectively, case:

Example: It can be easily seen by graphing the two probability distributions given below that F dominates G in the ørst degree.

$F(x) = 1 - e^{-x}$ if $x \succeq 0$

$F(x) = 0$ if $x \prec 0$
and
$G(x) = 1 - e^{-2x}$ if $x \succeq 0$
$G(x) = 0$ if $x \prec 0.$

Alternatively we can verify this fact by using the following Lemma:

Lemma: F dominates G in the ørst degree (FSD) in a given set of outcomes C if and only if

$$\int_C u(x)g(x)dx = \int_C u(x)dG(x)dx$$

for all strictlyincreasing utility functions $u(x)$.

This property is satisøed by all individuals who are not satiated.

Second Degree Deønition. F dominates G in the second degree **(SSD)** in a given set of outcomes C if and only if for a given x

$$\int_{-\infty}^{x} F(y)dy \geq \int_{-\infty}^{x} G(y)dy.$$

To understand the meaning of the SSD compare the areas under the curves.

An important implication of the SSD is the following:
Lemma. F dominates G in the second degree (SSD) if and only if

$$\int_c u(x)dF(x) \geq \int_c u(x)dG(x)$$

for all strictly increasing and concave utility functions $u(x)$.

The conditions of this lemma hold among all agents with increasing and concave u. This means that given a set of prospects, all such individuals are unanimous in their choice of the most preferred one.

New section – Empirical oberevations indicate that most people are risk- averse. The following deønitions intent to measure the strength of risk aversion of an individual:

Measure of absolute risk aversion:

$$-\frac{u''(x)}{u'(x)} \equiv R_a(x).$$

The absolute measure of risk aversion captures the curvature of the utility function. A related concept is the following:

Measure of relative risk aversion:

$$-x\frac{u''(x)}{u'(x)} \equiv R_r(x).$$

Intuitively, the more risk-averse is an individual the □more concave□ must be the utility function. This intuition is formalized in the above deønitions. From the properties of u, both measures (evaluated at some point x) are positive numbers.

Portfolio Theory-Interpretation of the Measure of Risk Aversion

Portfolio Suppose that their are two assets ; one risk-free which gives rate of return r and one risky asset with rate of return $\widetilde{r}$ (where $\widetilde{r}$ is a random variable (RV)). A risk averse individual has present wealth W_0 and invests a dollars in the risky asset. His /her future wealth will is:

$$\begin{aligned}\widetilde{W} &= (W_0 - a)(1+r) + a(1+\widetilde{r}) \\ &= W_0(1+r) + a(\widetilde{r} - r).\end{aligned}$$

It is assumed that the preferences of the individual are represented by a utility function that is concave and non-decreasing. The expected utility theory assumes that the individual will maximize the utility of future wealth; i.e.

$$E[u(\widetilde{W})] = E[u(W_0)(1+r) + a(\widetilde{r} - r)]$$

with respect to a.

Suppose there is an interior solution to the above problem. The ørst order condition (FOC) are:

$$E[u'(\widetilde{W})(\widetilde{r} - r)] = 0,$$

where $u' > 0$. Since $u' > 0$, this means that:
∘Prob $\{\widetilde{r} - r\} > 0$ for some values of the RV $\widetilde{r}$, and
∘Prob $\{\widetilde{r} - r\} < 0$ for some values of the RV $\widetilde{r}$.
i.e. the individual invests in the risky asset only if the risk premium on the interest rate is positive.

If $E[\tilde{r} - r] > 0$ then $\alpha > 0$

Proposition. Let u be concave. Then, $a > 0$ whenever $E[\widetilde{r} - r] > 0$,i.e. in order for an individual to invest in the risky asset, the risk-premium must be positive.

Proof: Suppose that the opposite holds, i.e. $a \leqslant 0$. Then, the FOC for a maximum becomes:

$$E[u'(W_0(1+r))(\widetilde{r} - r)] \leqslant 0,$$

This means that,

$$u'(W_0(1+r))E[\widetilde{r}-r] \leqslant 0,$$

or $E[\widetilde{r}-r] \leqslant 0$, since u' is positive. This last conclusion contradicts our assertion ~~that $a > 0$~~. that $E[\tilde{r}-r]>0$. Thus α cannot be negative

The converse of the above statement is also true; thus $a > 0$ if and only if $E[\widetilde{r}-r] > 0$. ↓ if $\alpha > 0$ then $E[\tilde{r}-r]>0$

In other words, the non-negativity condition asserts that if $a \geq 0$, the FOC. evaluated at $a=0$, must be non-positive. Note that this holds for only one risky asset.

One might ask the following question: What is the minimum risk premium which the individual requires in order to invest all of his/her initial wealth in the risky asset? To answer this consider the FOC: expand the term under the expectation sign in Taylor series around $W_0(1+r)$:

$$E[u'(\widetilde{W})(\widetilde{r}-r)] = E[u'(W_0(1+r)) + W_0 u''(W_0(1+r))(\widetilde{r}-r)^2 + R)] = 0,$$

or, by omitting the remainder, the term after the first equality sign becomes

$$u'(W_0(1+r))E[\widetilde{r}-r] + W_0 u''(W_0(1+r))E[(\widetilde{r}-r)^2] = 0,$$

and rearranging terms & dividing by $u'(\cdot)$ we obtain:

$$E[\widetilde{r}-r] = R_a(W_0(1+r))W_0 E[(\widetilde{r}-r)^2],$$

where

$$R_a = -\frac{u''(.)}{u'(.)}.$$

This expression gives the risk premium of a risky prospect and holds for □small risks□, i.e. for prospects with small variance. Note that, the higher the R_a the higher the minimum risk premium, $E[r-r]$, which is required to cause a to be equal toW_0 at the optimum. I.e. R_a measures the intensity of aversion to risk.

clearly ~~Note that~~ R_a depends on W_0, i.e. $R_a = R_a(W_0)$. We say the individual displays:

Definition

·**decreasing absolute risk aversion** (DARA) if $\frac{dR}{dW_0} < 0$,

·**increasing absolute risk aversion** (IARA) if $\frac{dR}{dW_0} > 0$,

·**constant absolute risk aversion** (CARA) if $\frac{dR}{dW_0} = 0$.

The signiøcance of these deønitions is the following: Suppose that, after an individual has invested a certain amount of initial wealth in the risky asset, his/her initial wealth increases. If the individual preferences display DARA he/she will invest in dollar terms less in the risky asset; i.e. the risky asset is a Giffen good. because The increase in initial wealth makes the person more averse to risk . This maybe because having

higher wealth he/she feels more secure, and not willing to risk future welfare by investing in the risky asset. By contrast in the case of DARA the individual will invest more dollars in the risky asset. Being richer, the person feels that he/she can a□ord to risk a greater part of initial wealth for the possibility of higher future welfare. In the case of CARA the individual will not invest any more or less than before in the risky asset; all the increase in initial wealth will be allocated to the risk-free asset. These results are summarized in the following theorem:

Theorem:

$$\begin{aligned}
\frac{dR_a(W_0)}{dW_0} &> 0 \text{ for all } W_0 \Longrightarrow \frac{da}{dW_0} < 0 \quad \text{(increasing risk aversion)}\\
\frac{dR_a(W_0)}{dW_0} &= 0 \text{ for all } W_0 \Longrightarrow \frac{da}{dW_0} = 0 \quad \text{(constant risk aversion)}\\
\frac{dR_a(W_0)}{dW_0} &< 0 \text{ for all } W_0 \Longrightarrow \frac{da}{dW_0} > 0 \quad \text{(decreasing risk aversion).}
\end{aligned}$$

It can be easily seen that non-decreasing absolute risk aversion implies that $u'''(.) > 0$.

Relative Risk Aversion Similar deønitions and properties are related to the relative measure of risk aversion which was deøned above as follows:

$$R_r(W_0) = W_0 R_a(W_0).$$

The di□erence is that now one is concerned with the proportion of initial wealth (rather than the absolute amount) invested in the risky asset. **Increasing relative risk aversion** means that an $x\%$ increase in initial wealth leads to a less than $x\%$ investment in the risky asset. In other words, the elasticity of demand of the risky asset is less than one. Similarly **decreasing (constant) relative risk aversion** implies that the elasticity of demand of the risky asset is greater than (equal to) one. The following theorem captures these results.

Theorem

$$\begin{aligned}
\frac{dR_r}{dW_0} &> 0 \Longrightarrow \frac{W_0}{a}\frac{da}{dW_0} < 1, \text{ (increasing relative risk aversion)}\\
\frac{dR_r}{dW_0} &= 0 \Longrightarrow \frac{W_0}{a}\frac{da}{dW_0} = 1, \text{ (constant relative risk aversion)}\\
\frac{dR_r}{dW_0} &< 0 \Longrightarrow \frac{W_0}{a}\frac{da}{dW_0} > 1, \text{ (decreasing relative risk aversion)}
\end{aligned}$$

Proof of DARA. From the ørst order conditions we obtain:

$$\frac{d\alpha}{dW_0} = \frac{E[u''(\widetilde{W_0})(\widetilde{r}-r)](1+r)}{-E[u''(\widetilde{W_0})(\widetilde{r}-r)^2}$$

where the singe of this expression depends on the sign of the denominator.

For $\widetilde{r} > r$, $a = W_0 > 0$ and, therefore, $\widetilde{W} > W_0(1+r)$, which implies that $R_a(\widetilde{W}) \leqslant R_a(W_0(1+r))$.

For $\widetilde{r} = r$, $\alpha = W_0 = 0$ and, therefore, $\widetilde{W} = W_0(1+r)$,which implies that $R_a(\widetilde{w}) = R_a(w_0(1+r))$.

For $\widetilde{r} < r$, $\alpha = W_0 > 0$ and therefore, $\widetilde{W} < W_0(1+r)$, which implies that $R_a(\widetilde{w}) \geq R_a(w_0(1+r))$.

Next multiplying both sides of the above expressions by $-u'(\widetilde{W})(\widetilde{r}-r)$:

$$-u'(\widetilde{W})R_a(\widetilde{W})(\widetilde{r}-r)\{>,=,<\}u'(\widetilde{W})R_aW_0(1+r)(\widetilde{r}-r),$$

respectively. Take expectations on both sides of the above expression and uses the FOC to derive the desired result.

The properties of absolute and relative risk aversion are local properties of utility functions. However, for some functions they are also global, i.e. they are satisøed in the whole range of deønition of the function. For example, power, quadratic, and exponential utility functions, satisfy these properties globally.

5 Mean-Variance Approach

5.1 Concepts from Probability Theory

Let $\widetilde{x} = \{x_1, x_2, ..., x_m\}$ and $\{p_1, p_2, ..., p_m\}$ denote the probability distribution of the random variable $\widetilde{x}$. We assume that the more important characteristics of risky assets are the mean (or, expected return,) and the variance of the distribution. Thus, we deøne the following terms:

Expectation

$$E[\widetilde{x}] = \Sigma p_i(x_i).$$

Variance

$$Var(\widetilde{x}) \equiv \sigma^2(\widetilde{x}) = E[(x - E[\widetilde{x}])^2] = \Sigma p_j(x_j - E[\widetilde{x}])^2$$

The last equality follows from the deønition of the expectation.

Let $\widetilde{x}_1$, $\widetilde{x}_2$, and $\widetilde{x}_3$ be three random variables. Then. the **Covariance** between two random variables $\widetilde{x}_1$, and $\widetilde{x}_2$:

$$Cov(\widetilde{x}_1, \widetilde{x}_2) \equiv \sigma(\widetilde{x}_1, \widetilde{x}_2) = E[(x_1 - E[\widetilde{x}_1])(x_2 - E[\widetilde{x}_2])].$$

$E[X_1 X_2] - E[X_1 E[X_2]]$
$- E[X_2 E[X_1]] + E[X_1]E[X_2]$
$= E[X_1 X_2] - 2E[X_1]E[X_2]$
$+ E[X_1]E[X_2]$
$= E[X_1 X_2] - E[X_1]E[X_2]$

Correlation coefficient r_{xy} between two random variables $\widetilde{x}$ and $\widetilde{y}$:

$$r_{xy} = \frac{Cov(\widetilde{x}, \widetilde{y})}{\sqrt{Var(\widetilde{x})Var(\widetilde{y})}}.$$

The correlation coe□ cient is such that $-1 \leqslant r_{xy} \leqslant 1$. When $r_{xy} = -1$ y is related to x as $y = a - bx$, and when $r_{xy} = 1$ as $y = a + bx$.

Their Properties

$$E[a\widetilde{x}] = aE[\widetilde{x}],$$

$$E[a + \widetilde{x}] = a + E[\widetilde{x}],$$

$$Var(\widetilde{x} + a) = Var(\widetilde{x}),$$

$$Var(a\widetilde{x}) = a^2 Var(\widetilde{x}),$$

$$E[\widetilde{x}_1 + \widetilde{x}_2] = E[\widetilde{x}_1] + E[\widetilde{x}_2]$$

$$Var(a\widetilde{x}_1 + b\widetilde{x}_2) = a^2 Var(\widetilde{x}_1) + 2abCov(\widetilde{x}_1, \widetilde{x}_2) + b^2 Var(\widetilde{x}_2),$$

$$Cov([a\widetilde{x}_1 + b\widetilde{x}_2], \widetilde{x}_3) = aCov(\widetilde{x}_1, \widetilde{x}_3) + bCov(\widetilde{x}_2, \widetilde{x}_3)$$

Returns of Risky Assets Consider a vector of n random variables $\widetilde{\mathbf{r}} = (\widetilde{r}_1, \widetilde{r}_2, ..., \widetilde{r}_n)$, each r_i representing the rate of return of n risky assets. Let p be a portfolio of these assets, and $\mathbf{w} = (w_1, w_2, ..., w_n)$ the vector of weights of these assets in the portfolio. The return of the portfolio is

$$\widetilde{r}_p = w_1\widetilde{r}_1 + w_2\widetilde{r}_2 + ... + w_n\widetilde{r}_n$$

$\widetilde{r}_1 = (r_{11}, r_{12})(P_1, 1-P_1)$
$\widetilde{r}_2 = (r_{21}, r_{22})(P_2, 1-P_2)$

and its expected return

$w_1[(r_{11}, r_{12})(P_1, 1-P_1)$
$+w_2[(r_{12}, r_{22})(P_2, 1-P_2)$

$$E[\widetilde{r}_p] = E[\sum_i w_i\widetilde{r}_i] = \sum_i w_i E[\widetilde{r}_i].$$

$\frac{1}{2}[P_1 r_{11} + (1-P_1) r_{11}]$
$+ \frac{1}{2}[P_2 r_{12} + (1-P_2) r_{21}]$
$=$

Similarly, the variance of the portfolio is

$w_1(r_{11} P_1 + r_{12} P_1) + w_2(r_{21} P_2 + r_{22}(1-P_2))$

$w_1 r_{11}, w_1 r_{12}, w_2 r_{21}$

$$Var(\widetilde{r}_p) \equiv \sigma^2(\widetilde{r}_p) = E[(\widetilde{r}_p - E[\widetilde{r}_p])^2].$$

For $n = 2$ the above expression is equal to:

$$\begin{aligned} & E[w_1^2(r_1 - E[\widetilde{r}_1])^2 + w_2^2(r_2 - E[\widetilde{r}_2])^2 + 2w_1w_2(r_1 - E[\widetilde{r}_1])(r_2 - E[\widetilde{r}_2]] \\ = \; & w_1^2 Var(\widetilde{r}_1) + w_2^2 Var(\widetilde{r}_1) + 2w_1w_2 Cov(\widetilde{r}_1, \widetilde{r}_2), \end{aligned}$$

The following diagram depicts the mean-variance relation of a portfolio of two risky assets as the proportion of wealth invested in the ørst asset (w_1) is increased.

The diagram below shows shape of the trade-o□ between mean and variance depending on the size if the correlation coe□ cient.

5.2 Foundations of the Mean-Variance

A particular version of Expected Utility is the Mean-Variance approach. It is assumed that utility is deøned over the mean and the variance of the distribution of returns of ønal wealth.

To motivate the mean-variance approach expand the Expected Utility functional in Taylor series around the expected end of period wealth:

It is a number

$$\begin{aligned} E[u(\widetilde{W})] &= E[u(E[\widetilde{W}])] + u'(E[\widetilde{W}])(\widetilde{W} - E[\widetilde{W}]) + \frac{1}{2}u''(E[\widetilde{W}])(\widetilde{W} - E[\widetilde{W}])^2 + R_3 \\ &= u(E[\widetilde{W}]) + \frac{1}{2}u''(E[\widetilde{W}])\sigma^2(\widetilde{W}) + R_3. \end{aligned}$$

where R_3 is the remainder of the expansion. If we assume that R_3 is small we can neglect it and have the Expected Utility depend on the mean and the variance of the distribution of ønal wealth

The validity of this approach relies the probability distribution of returns of the assets, and on the properties of the utility function.

Consider the ørst issue. Suppose that the probability distribution of assets can be completely described by the expected return and the variance of their returns. This is indeed the case when the returns of assets are multivariate normally distributed. In this case, the returns of a portfolio consisting of such assets is itself normally distributed, which implies that individuals will choose mean-variance e□ cient portfolios. These properties of the normal distribution is not shared by other distributions. For example, if assets have a lognormal distribution, the returns of portfolios made up from such assets are not lognormally distributed. One problem with the normal distribution is that it is unbounded from below. Thus, it cannot deal with topics such

Distribution of returns

pro

con

as limited liability and the meanigless of negative consumption. However, normality is only a sufficient condition for the validity of the mean-variance approach.

The function which agrees with the mean-variance approach is the quadratic utility function. With this function moments of order higher that the second one are zero, and the approach is valid for arbitrary distributions, because R_3. is indeed zero. However this function has some undesirable properties and leads to counter-intuitive results, as we will see later. For other functions (such as the power, the logarithmic and the exponential utility functions) moments higher than the second may play a significant role and cannot be safely neglected. In addition these functions are not defined for negative rates of return, and they cannot handle topics such as bankruptcies and unlimited liability. function

We will have to say more about the Mean-Variance approach in later chapters.

5.3 Portfolio Frontier

5.3.1 Two Risky Assets

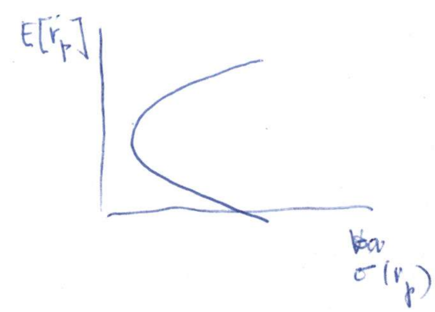
Consider a portfolio p of two assets with w proportion of the first asset and $1-w$ of the second asset. Construct a number of portfolios of the same two assets by letting w take different values between zero and one. Then, plot in a diagramme the mean and the variances of each of these portfolios. What you will derive is a diagramme like the following: (***********).

This is called the **feasible portfolio frontier**. Note that the for a certain range of expected returns the variance first increases and then decreases. In addition, there is one portfolio which has the smallest variance among all these portfolios; it is called the **minimum variance portfolio** (mvp). We wish to find the expression for the mvp.

The variance of a portfolio p consisting of two assets is

$$Var(\widetilde{r}_p) \equiv w^2\sigma^2(\widetilde{r}_1) + 2\overset{w(1-w)}{Cov}(\widetilde{r}_1, \widetilde{r}_2) + (1-w)^2\sigma^2(\widetilde{r}_2)$$

To find the minimum of mvp we differentiate the above expression with respect to w and set the derivative equal to zero; i.e.



$$\frac{dVar(\widetilde{r}_p)}{dw} = 0$$

Solving for w we obtain:

$$w^* = \frac{\sigma^2(\widetilde{r}_2) - Cov(\widetilde{r}_1, \widetilde{r}_2)}{\sigma^2(\widetilde{r}_1) + \sigma^2(\widetilde{r}_2) - 2Cov(\widetilde{r}_1, \widetilde{r}_2)}$$

The expectation of the mvp is

$$E[\widetilde{r}_{\underset{mvp}{\cancel{p}}}] = w^*E[\widetilde{r}_1] + (1-w^*)E[\widetilde{r}_2]$$

By combining the last two expressions we obtain another expression for w^*:

$$w^* = \frac{E[\widetilde{r}_p] - E[\widetilde{r}_2]}{E[\widetilde{r}_1] - E[\widetilde{r}_2]}.$$

Next, suppose the individual wishes to ønd the portfolio which has the smallest variance for a **given** rate of return among all the portfolios described above. In the case of two assets the problem is null. There is a unique combination of the two assets which yields a speciøed rate of return and, therefore , there is a unique variance associated with it. The problem becomes interesting in the case of many risky assets.

5.3.2 Many risky Assets

Suppose that there are n risky assets. Let $\widetilde{\mathbf{r}} = (\widetilde{r}_1, \widetilde{r}_2, ..., \widetilde{r}_n)$ be the returns of these assets, and $\mathbf{w} = (w_1, w_2, ..., w_n)$ their weights and in some portfolio p. The expected return of p is:

$$E[\widetilde{r}_p] = \sum_i w_i E[\widetilde{r}_i],$$

and its variance is:

$$Var(\widetilde{r}_p) = \mathbf{w}' V \mathbf{w} = \sum_i \sum_j w_i w_j \sigma(\widetilde{r}_i, \widetilde{r}_j)$$

where $\sigma(\widetilde{r}_i, \widetilde{r}_j)$ is an alternative notation for the covariance between the ith and the jth assets. The variance-covariance matrix between two portfolios, say p and q, is deøned to be:

$$Cov(\widetilde{r}_p, \widetilde{r}_q) = \mathbf{w}_p' \mathbf{V} \mathbf{w}_q.$$

Consider all possible portfolios that can be constructed with these assets and search for the one which has the smallest variance for a **given** rate of return. To ønd it consider the following minimization problem:

minimize

$$\frac{1}{2}\mathbf{w}_p' V \mathbf{w}_p = \frac{1}{2}\sum_i \sum_j w_i w_j \sigma(\widetilde{r}_i, \widetilde{r}_j),$$

? see next page

subject to:

$$E[\widetilde{r}_p] = \sum_i w_i E[\widetilde{r}_i], \text{ and } \sum_i w_i = 1.$$

required

Here V is the variance-covariance matrix of the assets. By varying the rate of return one can construct a set of portfolios (called the **portfolio frontier** or **minimum**

variance opportunity set) which yield the smallest variance for di□erent expected (or, required) rates of return. The portfolio frontier is the envelope of the feasible opportunity set. To ønd the set of portfolios which belong to the porotfolio fronftier, form the Lagrangean expression:

$$L = \frac{1}{2}\sum_i\sum_j w_i w_j \sigma(\widetilde{r}_i, \widetilde{r}_j) - \lambda_1[E[\widetilde{r}_p] - \sum_i w_i E[\widetilde{r}_i]] - \lambda_2[\sum_i w_i - 1]$$

without specifying the required rate of return.

di□erentiate with respect to $w_i, i = 1, 2, ..., n,\ \lambda_1, \lambda_2,$ and set each derivative equal to zero. Then, solve the system of $n+2$, equations which result . A general expression for the solution is rather messy, but the shape of the portfolio frontier is the same as in the case of two assets. Here we state the following result:

Proposition. Let the variance-covariance matrix V be positive deønite. For any portfolio p on the portfolio frontier there exist two column vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{w}_p = \mathbf{a} + \mathbf{b}E[\widetilde{r}_p],$$

where the components of $\mathbf{a}$ and $\mathbf{b}$ depend on the expected returns of the assets and their variance-covariance matrix.

The proof follows directly by solving the ørst order conditions of the above optimization problem. It can be shown that the portfolio frontier drawn with di□erent expected returns and the corresponding minimum standard deviations is a parabola.

In the diagram which follows the parabola is the minimum variance opportunity set of portfolios consisting of di□erent combinations of several risky assets.

give the diagram

Portfolio A is the **minimum variance portfolio**, and B is some asset, or portfolio, on the frontier. Assets B and C are not on the frontier and it may appear that they will not be traded because they have too much risk for the return they yield. However, as it will be seen, in equilibrium the price of an asset reÆects not its total risk but the part of the risk which relates to the economy as a whole.

5.4 Properties of the frontier

Below we state some properties of the portfolio frontier.

Fact 1. A linear combination of two portfolios on the portfolio frontier is also on the frontier; i.e. the entire portfolio frontier can be generated by knowing only two distinct frontier portfolios.

Proof. Consider two distinct portfolios on the frontier, p_1, p_2 and let q be any frontier portfolio. Then, there is a unique number α such that

$$E[\widetilde{r}_q] = \alpha E[\widetilde{r}_{p1}] + (1-\alpha)E[\widetilde{r}_{p2}].$$

Make a new portfolio which consists of p_1 with weight α, and p_2 with weight $(1-\alpha)$; i.e.

Extra Class
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time Friday 3:00 - 4:00

$$
\begin{aligned}
W_q = \alpha \mathbf{w}_{p1} + (1-\alpha)\mathbf{w}_{p1} &= \alpha(\mathbf{a}+\mathbf{b}E[\widetilde{r}_{p1}]) + (1-\alpha)(\mathbf{a}+\mathbf{b}E[\widetilde{r}_{p2}]) \\
&= \mathbf{a}+\mathbf{b}(\alpha E[\widetilde{r}_{p1}] + (1-\alpha)E[\widetilde{r}_{p2}]) \\
&= \mathbf{a}+\mathbf{b}E[\widetilde{r}_q] = \mathbf{w}_q.
\end{aligned}
$$

Thus, the weights of q are a linear combination of those of p_1, and p_2.

Portfolio mvp is the minimum variance portfolio. Portfolios which yield expected return greater that the expected return of mvp are called e□cient portfolios, and the part of the frontier which consists of e□cient portfolios is called the **efficiency frontier**. ✓ ✓

Fact 2. The set of e□cient portfolios is a convex set. ✓

Proof. Let $\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m$ be ~~frontier~~ efficient ok portfolios. Consider a ~~linear~~ convex combination of these portfolios:

$$\sum_{i=1}^{m} \alpha_i \mathbf{w}_i$$

for some constants $\alpha_1, \alpha_2, ..., \alpha_m$, such that $\sum_i^m \alpha_i = 1$. Since w□s are frontier portfolios, the above expression is equal to

$$
\begin{aligned}
&\sum_{i=1}^{m} \alpha_i(\mathbf{a}+\mathbf{b}E[\widetilde{r}_i]) \\
&= \mathbf{a}+\mathbf{b}(\sum_{i=1}^{m} \alpha_i E[\widetilde{r}_i]):
\end{aligned}
$$

→ Since a lin. combin. of frontier portfolios is a frontier portfolio it follows that. i.e.

incomplete. See H+L. p 69

Thus any ~~linear~~ convex combination of frontier portfolios is ~~on the frontier~~ an efficient portfolio because.

Fact 3 The covariance between any portfolio p and the minimum variance portfolio (mvp) is equal to the variance of mvp; i.e.

$$Cov(\widetilde{r}_p, \widetilde{r}_{mvp}) = Var(\widetilde{r}_{mvp}).$$

Proof: Consider a portfolio q on the frontier consisting of p and mvp with weights w and $1-w$. Its variance is

$$Var(\widetilde{r}_q) = w^2\sigma^2(\widetilde{r}_p) + 2w(1-w)Cov(\widetilde{r}_p, \widetilde{r}_{mvp}) + (1-w)^2\sigma^2(\widetilde{r}_{mvp}).$$

$2w\sigma^2(\tilde{r}_p) + 2(1-w)Cov(\tilde{r}_p, \tilde{r}_{mvp}) - 4w Cov(\tilde{r}_p, \tilde{r}_{mvp}) \mp 2(1-w)\sigma^2(\tilde{r}_{mvp})$

Being a frontier portfolio, q must have minimum variance; i.e.

$$\frac{dVar(\widetilde{r}_q)}{dw} = 0,$$

and since the mvp has minimum variance among all portfolios on the frontier, it must be that $w = 0$. Carrying out these calculations gives the expression in Fact 3.

As it will explained, the unanimity property of shareholders does not hold here without further assumptions.

5.5 Portfolio Frontier with a Risk-Free Asset

5.5.1 A Risky and a Risk-Free Asset

Suppose that there is one risky asset with rate of return $\widetilde{r}$ and one risk-free asset with rate r_f. The expected return of a portfolio p with weight w of the risky asset is:

$$E[\widetilde{r}_p] = wE[\widetilde{r}] + (1-w)r_f,$$

and its variance is:

$$Var(r_p) = w^2 Var(\widetilde{r}),$$

which equals the variance of the risky asset. By combining the two expressions we obtain the equation of the frontier:

diagram

$$E[\widetilde{r}_p] = r_f + \frac{E[\widetilde{r}] - r_f}{\sigma(\widetilde{r})}\sigma(\widetilde{r}_p).$$

In this case the portfolio frontier is a straight line; r_f is the intercept, and $\frac{E[\widetilde{r}]-r_f}{\sigma(\widetilde{r})}$ is its slope. The slope shows the rate at which the individual is willing to substitute risk for return. To see this consider the derivatives of the mean and the variance of the portfolio as w changes. #

This holds in general even if ∃ no risk free asset

$$\frac{dE[\widetilde{r}_p]}{dw} = E[\widetilde{r}] - r_f,$$

$$\frac{d\sigma(\widetilde{r}_p)}{dw} = \sigma(\widetilde{r}).$$

By combining these expressions we obtain:

$$\frac{dE[\widetilde{r}_p]}{d\sigma(\widetilde{r}_p)} = \frac{dE[\widetilde{r}_p]/dw}{d\sigma(\widetilde{r}_p)/dw} = \frac{E[\widetilde{r}] - r_f}{\sigma(\widetilde{r})},$$

which gives the trade-o□ between risk and return. This is the slope of the frontier. Along the frontier the portfolio contains proportion w of the risky asset and $1-w$ of the risk-free one. If $0 \leqslant w \leqslant 1$ the individual is a lender, because he/she does not invest all of his/her wealth in the risky asset. For $w > 1$ he/she borrows in order to invest enough in the risky asset. For $w < 0$ he/she lends more than his/her wealth by selling short the risky asset.

[Draw a diagramme showing the equilibrium of an individual with only risky assets. Assune that the agent is risk averse. Discuss the implications]

We assume that the utility functions of an individual is concave and increasing, and the returns of the risky securities are multivariate normally distributed. Under

these conditions it is easy to show that each individual has indifference curves for risk and return which are upward slopping; for a risk-averter return is a "good" and risk is a "bad". Two indifference curves between risk and return have been added to the opportunity set. The point of tangency between the opportunity set and an indifference curve is the optimum portfolio of the individual. The slope of the opportunity set (or, the marginal rate of transformation) indicates the rate at which the individual can substitute risk for return, given the assets which are traded. On the other hand the slope of the indifference curve (or, the marginal rate of substitution) is the rate at which he/she is willing to do so i.e.

$$MRS = MRT.$$

Note that given the shape of the indifference curves any portfolio on the frontier is an efficient one.

5.5.2 Many Risky Assets and a Risk-Free Asset

Suppose that there are n risky and one risk-free asset. The above optimization problem of the previous subsection becomes:
minimize

$$\frac{1}{2}\mathbf{w}^T V \mathbf{w} = \frac{1}{2}\sum_i\sum_j w_i w_j \sigma(\widetilde{r}_i, \widetilde{r}_j),$$

subject to

$$E[\widetilde{r}_p] = \sum_i w_i E[\widetilde{r}_i] + (1 - \sum_i w_i) r_f.$$

The optimization yields a portfolio frontier which, as it turns out, has the same expression as in the case of one risky asset. The difference is that in this case the role of the "risky asset" is played by portfolio M (see Figure ***) which lies on the frontier of only-risky-assets. It expression is:

$$E[\widetilde{r}_p] = r_f + \frac{E[\widetilde{r}_m] - r_f}{\sigma(\widetilde{r}_m)}\sigma(\widetilde{r}_p).$$

The frontier is a straight line with intrecept r_f and slope $\frac{E[\widetilde{r}_m]-r_f}{\sigma(\widetilde{r}_m)}$. All portfolios along the frontier are efficient; they share the property that they have minimum variance for a given expected return. Fron Figure *** note that the frontier is tangent to the the frontier of only-risky-assets at point M. If it was not, one could find portfolios with higher expected return for the same level of risk. It follows that point M is unique. [Draw a diagramme showing the equilibrium of an individual with only risky assets. Assune that the agent is risk averse. Add explanation fot risk averse. Discuss the implications]:(Samae interpretation as in the case of all risky assets).

Portfolios along the straight line consist of the risk-free asset and portfolio of risky assets M. This is called the **two fund separation property** of the frontier.

5.6 Market equilibrium

Our purpose is to be able to characterize the properties of security prices in equilibrium.

In ønancial markets there are traded numerous securities and they are all held by some investor. The law of one price, which is assumed to hold in equilibrium, requires all securities (or all portfolios of securities) which have the same distribution of returns must have the same price. This imposes some constraints on the prices of securities. Even thought we know the risk-return characteristics of a security, we cannot determine its equilibrium price by this information alone. [Add the diagramme showing the opportunity frontier and the risk-return combination of each asset. Explain whiassets with same return than a frontier portfolio but higher risk cae traded in the market]. In equilibrium, security prices adjust so that there is no excess demand for any one security. Thus, the price of each security reÆects the demand and, thus, the risk-return characteristics of all other securities. To take this interdependency into account we must know the joint distribution of returns of all traded securities. In the mean-variance approach, this requires that we know the variance-covariance matrix of all securities.

Suppose that there are N individuals trading in the market, j=1,2,...,N, and n risky securities, i=1,2,...,n. The jth individual invests a proportion w_{ij} of his/her initial wealth in the ith security. We assume that all future states have a positive probability to occur. Furthermore assume that individuals have homogeneous expectations, i.e. all individuals know the probability of each state to occur.

We deøne as the **market portfolio** the portfolio which consists of all the securities held by the market participants. In equilibrium the sum of initial wealth of the individuals must equal the total market value of these assets: $\sum_j W_j = \sum_i V_i$, where $V_i = \sum_j w_{ij} W_j$ is the market value of the ith security. Then, the weight of the ith securities in the marker portfolio must be (see notes on a previous version of notes used last year): (Also $W_m = \sum_j W_j$)

$$w_{mi} = \frac{V_i}{\sum_i V_i} = \frac{V_i}{\underbrace{\sum_j W_j}_{W_m}} = \frac{\sum_j w_{ij} W_j}{\underbrace{\sum_j W_j}_{W_m}} = \sum_j \cancel{w_{ij} \frac{W_j}{\sum_j W_j}} = \sum_j w_{ij} \frac{W_j}{W_m}$$

that is the weights of the market portfolio are a convex (and, thus, linear) combination of the portfolio weights of the individuals.

We assume that the returns of the risky securities are multivariate normally distributed and, thus, portfolios consisting of these assets are also multivariate normally distributed. In addition, the utility functions of all individuals are concave. Then, individuals prefer frontier portfolios. In addition, form Fact 1 on the properties of the frontier, a linear combination of any other two frontier portfolios is also a frontier

m. frontier portfolio

portfolio. Therefore, the market portfolio (which is a convex combination of individual portfolios, as we have seen above), is a frontier portfolio. Furthermore, if the utility functions of all individuals are monotone increasing their optimal portfolios are e□cient. It follows from Fact 2 that the market portfolio is also e□cient.

↓ efficient

5.6.1 Capital Market Line with a Risk-Free Asset

Now consider the case of an additional risky asset. We know that the e□ciency frontier of each individual is a straight line and that the optimum portfolios of risk averse individuals are e□cient. Thus, two-fund separation holds for each individual. Suppose that individuals have homogeneous expectations, i.e. they all agree on the probabilities of returns of the risky assets. This additional assumption has strong implications. It means that all individuals perceive the same portfolio frontier of only-risky-assets. It follows that with a risk-free asset the straight line e□ciency frontiers of all individuals have the same slope, the same intercept, and the same tangency point on the frontier on the only-risky-assets frontier. In other words, individuals perceive the same e□cient set for all assets, and the market portfolio is unique. Thus, two-fund separation holds for the whole market. *

To put it di□erently, at equilibrium a single tangency portfolio (i.e the market portfolio) is found such that two-fund separation holds, and all risky assets are held according to their value weights in the market portfolio. Thus,the Fisher separation theorem is still valid.

Now the set of e□cient portfolios can be written as:

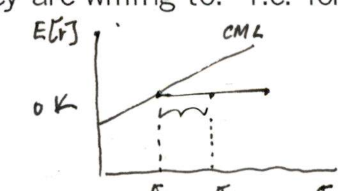
$$E[\widetilde{r}_p] = r_f + \frac{E[\widetilde{r}_m] - r_f}{\sigma(\widetilde{r}_m)}\sigma(\widetilde{r}_p).$$

The CML consists of efficient assets where $E[\tilde{r}_j] = a\,E[\tilde{r}_m] + (1-a)\,r_f$. Assets on CML have no unsystematic risk; all the risk is contributed by the mkt portfolio i.e. they are perfectly correlated with m. Assets w/ non-systematic risk and a given exp. return lie on the horizontal line in the figure below. The distance $\sigma_2 - \sigma_1$ measures the non-systematic risk.

where $E[\widetilde{r}_m]$ and $\sigma(\widetilde{r}_m)$ are the expected return and the standard deviation of the market portfolio. This is called the Capital Market Line (CML). (Bold)

The slope is the rate at which all investors can substitute risk for return by participating in the capital market. The tangency between the e□ciency frontier and an indi□erence curve of each individual suggest that in equilibrium all participants in the market can to substitute risk for return at the rate they are willing to. I.e. for any two individuals (the $ith$and the jth):

$$MRS_i = MRS_j = \frac{E[r_m] - r_f}{\sigma(r_m)}.$$ ok



6 The Capital Asset Pricing Model

The discussion in the previous section allows us to characterize the prices of either individual assets, or of portfolios of assets; which was our initial intent.

Consider asset i traded in the market. In equilibrium its price must have adjusted so that the asset is part of the market portfolio; otherwise it would not be traded.

Make a new portfolio p consisting of asset i (with weight α_i) and the market portfolio M (with weight ~~α_i~~ $1-\alpha_i$).

$$E[\widetilde{r}_{p}] = \alpha_i E[\widetilde{r}_i] + (1-\alpha_i)E[\widetilde{r}_m],$$

$$\sigma(\tilde{r}_p) = (\alpha_i^2\sigma_i^2(\widetilde{r}_i) + 2\alpha_i(1-\alpha_i)\sigma(\widetilde{r}_i,\widetilde{r}_m) + (1-\alpha_i)^2\sigma^2(\widetilde{r}_m))^{\frac{1}{2}}.$$

Note that if $a > 0$, p contains more of the asset i than what the market portfolio does; i.e. if someone holds p there is an excess demand in the market for asset i. To see the implications of di□erentiate the above expressions with respect to α_i and evaluate the derivatives at $\alpha_i = 0$. Why at a_i equal to zero? Because as it has been said the market portfolio M includes all assets which are traded in equilibrium (and, thus, asset i) and if p contains any amount of asset i greater than zero, it must contain the excess (ie. where $D_i = S_i$) demand for this asset. But we know that in equilibrium the excess demand for all assets is zero.

The di□erentiations give:

$$\frac{dE[\tilde{r}_p]}{d\alpha_i} = E[\tilde{r}_i] - E[\tilde{r}_m]$$

and

$$\frac{d\sigma(r_p)}{d\alpha_i} = \frac{1}{2}[\sigma^2(\tilde{r}_p)]^{-\frac{1}{2}}[2\sigma(\widetilde{r}_i,\widetilde{r}_m) - 2\sigma^2(\widetilde{r}_m)] = \frac{\sigma(\widetilde{r}_i,\widetilde{r}_m) - \sigma^2(\widetilde{r}_m)}{\sigma(\widetilde{r}_m)}$$

By dividing the two expressions we obtain the risk- return trade-o□:

$$\frac{dE[\widetilde{r}_p]}{d\sigma(\widetilde{r}_p)} = \frac{dE[\widetilde{r}_p]/d\alpha_i}{d\sigma(\widetilde{r}_p)/d\alpha_i} = \frac{E[\widetilde{r}_i] - E[\widetilde{r}_m]}{\sigma(\widetilde{r}_i,\widetilde{r}_m) - \sigma^2(\widetilde{r}_m)}\sigma(\widetilde{r}_m).$$

This is a relation between asset i and the market portfolio (M) which must hold in equilibrium; it is the slope of the frontier at M. In the last section we showed that the Capital Market Line is also an equilibrium relationship where all assets are held according to their market value weights. Equating this expression with the one of the CML we obtain:

$$\frac{E[\widetilde{r}_i] - E[\widetilde{r}_m]}{\sigma(\widetilde{r}_i,\widetilde{r}_m) - \sigma^2(\widetilde{r}_m)} = \frac{E[r_m] - r_f}{\sigma(r_m)},$$

or,

$$
\begin{aligned}
E[\widetilde{r}_i] &= r_f + \frac{E[\widetilde{r}_m] - r_f}{\sigma^2(\widetilde{r}_m)}\sigma(\widetilde{r}_i, \widetilde{r}_m) \\
&= r_f + \beta_i(E[\widetilde{r}_m] - r_f) = (1 - \beta_i)r_f + \beta_i E[\widetilde{r}_m],
\end{aligned}
$$

where,

$$\beta_i \equiv \frac{\sigma(\widetilde{r}_m, \widetilde{r}_i)}{\sigma^2(\widetilde{r}_m)}.$$

The above expression is called the Capital Asset Pricing Model (CAPM), and its graph, the **Security Market Line (SML)**. Thus, in equilibrium, the rate of returns one expectes (or, requires) from any asset is equal to the risk-free rate (r_f), plus a risk premium ($E[\widetilde{r}_m] - r_f$). The latter consists of the exess returns of the market portfolio times a constant. The constant, called **beta**, depends on the covariability of the asset with the market portfolio, and it can be considered as the quantity of risk which the asset contributes to the market portfolio.

The beta for the market portfolio is one:

$$\beta_m = \frac{\sigma(\widetilde{r}_m, \widetilde{r}_m)}{\sigma^2(\widetilde{r}_m)} = 1,$$

and that of a risk-free asset is zero.

[Modify the diagram]

Interpretation and Properties of the CAPM The expression for the SML says that in equilibrium the appropriate measure of risk for a single asset is its beta; the risk premium of an asset is beta times the risk premium of the market portfolio.

$$E[\widetilde{r}_j] - r_f = \beta_i(E[\widetilde{r}_m] - r_f),$$

Thus, in equilibrium, the price of every risky asset must adjust so as the rate of return which investors require falls exactly on the Security Market Line (SML). Securities which do not lie on the e□ ciency frontier lie exactly on the SML.

In another way the above expression for the CML ban be written as

$$E[\widetilde{r}_j] = r_f + \lambda Cov(\widetilde{r}_j, \widetilde{r}_m)$$

where,

$$\lambda \equiv \frac{E[\widetilde{r}_m] - r_f}{Var(\widetilde{r}_m)}$$

Here $Cov(\widetilde{r}_j, \widetilde{r}_m)$ is the contribution of asset j to the risk of the market portfolio, and λ is interpreted as the market price of (exess) returns per unit of risk.

To understand the interpretation of beta consider the following. If the the theory of the CAPM is correct, the following empirical relationship must hold for any asset j and the market portfolio:

$$\widetilde{r}_j = a + b\widetilde{r}_m + \widetilde{\varepsilon}_j$$

where the error term $\widetilde{\varepsilon}_j$ has zero mean and zero covariance with the returns of the market portfolio $\widetilde{r}_m$.. The expected return of $\widetilde{r}_j$ is:

$$E[\widetilde{r}_j] = a + b_j E[\widetilde{r}_m].$$

The expression of the CAPM, written in a di□erent way, is

$$E[r_j] = (1 - \beta_j) r_f + \beta_j E[\widetilde{r}_m]$$

This means that:

$$a = (1 - \beta_j) r_f, \text{ and } b_j = \beta_j.$$

Similarly the variance (or, total risk) of $\widetilde{r}_j$ is

$$\begin{aligned} \sigma^2(\widetilde{r}_j) &= b_j^2 \sigma^2(\widetilde{r}_m) + \sigma^2(\widetilde{\varepsilon}_j) \\ &= \beta_j^2 \sigma^2(\widetilde{r}_m) + \sigma^2(\widetilde{\varepsilon}_j). \end{aligned}$$

because the covariance bertwøen the market portfolio and the error term is zero. The last expression says that, the total risk of the asset consists of two parts: one which is priced by the capital market (systematic risk), and one that is totally random (unsystematic risk), i.e.

Total Risk= Systematic Risk + Unsystematic Risk.

The sysrtematic risk relates to risks inherent in the functioning of the whole economy, the unsustematic risk depends on various disturbances which are independent of the economy. The latter can be diversiøed away by investing in a portfolio with many assets, the former is inescapable and in order to avoid it investors are prepared to pay a preminum. Thus, one cannot use the variance of an asset as a measure of its risk because part of it can be diversiøed away.

An important property of the CAPM is that the risk of individual assets are linearly additive when they are combined into portfolios; this follows from the linearity of covariances. E.g. for two assets i and j :

$$\beta_p = w_i \beta_i + w_j \beta_j$$

The Prices of Securities The relation between the present price of an asset P_j, its futere anticipated price $\widetilde{P}_j$, and its rate of return $\widetilde{r}_j$, is known to be:

$$\widetilde{r}_j = \frac{\widetilde{P}_j - P_j}{P_j}.$$

By taking expectations and using the expression for the CML we obtain□

$$\begin{aligned} E[\widetilde{r}_j] &= \frac{E[\widetilde{P}_j] - P_j}{P_j} = r_f + (E[\widetilde{r}_m] - r_f)\frac{\sigma(\widetilde{r}_m, \widetilde{r}_j)}{\sigma^2(\widetilde{r}_m)} \\ &= r_f + \lambda Cov(r_j, r_m), \end{aligned}$$

and solving for P_j:

$$P_j = \frac{E[\widetilde{P}_j]}{1 + r_f + \lambda Cov(\widetilde{r}_j, \widetilde{r}_m)} \qquad \surd$$

This formula is used to derive the prices of securuties in single period models. Todays price is the discounted value the of expeted future price. It is called the risk-adjusted valuation method because of the form of the denominator

Applications

$$E[\widetilde{r}_j] = r_f + \beta_j(E[\widetilde{r}_m] - r_f)$$

[

$$\begin{aligned} E[\widetilde{r}_i] &= a_i + b_i E[\widetilde{F}] \\ \sigma^2(\widetilde{r}_i) &= b_i^2 \sigma^2(\widetilde{F}) + \sigma^2(\widetilde{\varepsilon}_i) \\ \sigma(\widetilde{r}_i, \widetilde{F}) &= b_i b_j \sigma^2(\widetilde{F}) \\ b_i &= \frac{cov(\widetilde{r}_i, \widetilde{F})}{\sigma^2(\widetilde{F})} \end{aligned}$$

$\widetilde{r}_j]$ is the required rate of return on equity. In other words, it is the cost of equity to the ørm. X

Consider various projects k and l:

Project k has a higher return (r_k, b_k) but its required rate of return is even higher.If the managers want the project to earn the same or an even higher rate of return as theørm, the project would be accepted. But, the market requires a rate of return. $E[\widetilde{r}_k]$ is greater than r_k for the given risk of the project. According to these criteria, the project is rejected.

6.1 No Risk Free Asset

The risk-free asset and the market portfolio are two well deøned mutual funds. The expected return of any other asset can be expressed as linear combination of of the returns of these two funds. When there is no risk-free asset the market portfolio cannot be uniquely determined. Then, our goal is to ønd two portfolios which can be used as mutual funds and, in addition, have some of the properties of the market portfolio and the risk-free asset.

It turns out that one mutual fund can be any frontier portfolio (except the mvp). The second mutual fund can be constructed from the ørst one by going through the following steps: will be shown that

1. Take an arbitrary portfolio p on the frontier. Below p will play the role of the □market portfolio□.

2. Find all the portfolios which have a zero covariance with portfolio p. Any such portfolio z must have zero beta because:

$$\beta_z = \frac{Cov(\tilde{r}_p, \tilde{r}_z)}{\sigma^2(\tilde{r}_p)} = 0.$$

All these portfolios have the same the same systematic risk ($\beta = 0$) and, therefore, the same expected return.

3. Form among those portfolios choose the one which lies on the frontier because it has the smalest variance.

The weights of this particular portfolio can be found from the following minmization problem:

$$\begin{aligned}
\text{minimize} \quad \sigma^2(\tilde{r}_z) &= \frac{1}{2}\mathbf{w}_z'\mathbf{V}\mathbf{w}_z \\
\text{subject to } \mathbf{w}_z'\mathbf{V}\mathbf{w}_p &= 0 \\
\mathbf{e}'\mathbf{w}_z &= 1
\end{aligned}$$

where e is a column vector of ones. Call this the **zero covariance minimum variance** portfolio of p and denote it by $z(p)$. It can be easily shown that if p is an e□cient portøoio, $z(p)$ is ine□cient, i.e. $E[\tilde{r}_{z(p)}] < E[\tilde{r}_{mvp}]$: It has zero covariance with p (just as the risk-free asset has zero covariance with the market portfolio).

4. Construct a straight line joins $E[\tilde{r}_{z(p)}]$ and p. To ønd the slope of this line make a new portfolio q with a % in the portfolio p and $(1-a)$% in $z(p)$.

The expectation and the standard deviation of q are:

$$\begin{aligned}
E[\tilde{r}_q] &= aE[\tilde{r}_p] + (1-a)E[\tilde{r}_{z(p)}] \\
\sigma(\tilde{r}_q) &= [a^2\sigma^2(\tilde{r}_p) + 2a(1-a)Cov(\tilde{r}_p, \tilde{r}_{z(p)}) + (1-a)^2\sigma^2(\tilde{r}_{z(p)})]^{\frac{1}{2}},
\end{aligned}$$

respectively. Note that the covariance in the last expression is zero by construction.

5. Di□erentiate these expressions with respect to a and evaluate the derivatives at $a = 1$. The result is:

$$\frac{\partial E[\widetilde{r}_q]}{\partial a} = E[\widetilde{r}_p] - E[\widetilde{r}_{z(p)}]$$
$$\frac{\partial \sigma(\widetilde{r}_q)}{\partial a} = \sigma(\widetilde{r}_p)$$

(Why evaluate the derivatives at $a = 1$? Because we want portfolio p to mimic the market portfolio, which happens when investors allocates all of their wealth in this portfolio.)

6. Finally, take the ratio of the last two expressions

$$\frac{\frac{\partial E[\widetilde{r}_q]}{\partial a}}{\frac{\partial \sigma(\widetilde{r}_q)}{\partial a}} = \frac{E[\widetilde{r}_p] - E[\widetilde{r}_{z(p)}]}{\sigma(\widetilde{r}_p)}$$

This is the slope of the line and shows the risk-return trade-o□ in equilibrium. The equation of the line is:

$$E[\widetilde{r}_q] = E[\widetilde{r}_{z(p)}] + \frac{E[\widetilde{r}_p] - E[\widetilde{r}_{z(p)}]}{\sigma(\widetilde{r}_p)}\sigma(\widetilde{r}_q),$$

where $E[\widetilde{r}_{z(p)}]$ is the intercept. This expression looks very much like the CML.

We can think of portfolio p as the market portfolio, and rename it m for easy reference. Now it is easy to show that the expected return of any asset, whether it lies on the frontier or not, must be a linear combination of the rate of return of m (the □market portfolio□) and of its zero-beta portfolio $z(m)$. By To verify this follow the same procedure as in the derivation of the CAPM with a risk-free asset. You will see that in equilibrium the slope of the risk-return trade-o□ of any asset j evaluated at M is:

$$\frac{\frac{\partial E[\widetilde{r}_j]}{\partial a}}{\frac{\partial \sigma(\widetilde{r}_j)}{\partial a}} = \frac{E[\widetilde{r}_j] - E[\widetilde{r}_m]}{\sigma(\widetilde{r}_m, \widetilde{r}_j) - \sigma^2(\widetilde{r}_m)}\sigma(\widetilde{r}_m)$$

In this section we have derived two expressions for the trade-o□ between risk and return. By equating them we obtain:

$$\frac{E[\widetilde{r}_m] - E[\widetilde{r}_{z(m)}]}{\sigma(\widetilde{r}_m)} = \frac{E[\widetilde{r}_j] - E[\widetilde{r}_m]}{\sigma(\widetilde{r}_m, \widetilde{r}_j) - \sigma^2(\widetilde{r}_m)}\sigma(\widetilde{r}_m)$$

which, after simpliøcation, becomes:

$$E[\widetilde{r}_j] = (1 - \beta_j)E[\widetilde{r}_z] + \beta_j E[\widetilde{r}_m]$$

where

$$\beta_j = \frac{Cov(\widetilde{r}_j, \widetilde{r}_m)}{\sigma^2(\widetilde{r}_m)}$$

which is the securitie smarket line. Note that the choice of p is arbitrary. The fact that $z(p)$ is ine□cient means that some assets must be sold short. The above proceedere assumes that there are no restrictions on short selling. If such restrictions exist, then the linear CAPM is not valid.

7 Arbitrage Pricing Theory

7.1 Diversifiable and Non-diversifiable Risk

We will show that part of the portfolio risk can be diversiøed away by investing in a large number of securities. However, the total risk (namely, the part that is related to the economy-wide risks) can never be reduced to zero, no matter how much we diversify.

Consider a very simple model. Suppose that there are n assets, $i = 1, 2, ..., n$, and that the rate of returns $\widetilde{r}_i$ of asset i are a□ected by some factor F in a linear fashion,

$$\widetilde{r}_i = a_i + b_i\widetilde{F} + \widetilde{\varepsilon}_i$$

where.

$$E[\widetilde{\varepsilon}_i] = 0,\ E[\widetilde{\varepsilon}_i, \widetilde{\varepsilon}_j] = 0,\ E[(\widetilde{F} - E[\widetilde{F}]), \widetilde{\varepsilon}_i] = 0$$

for all i, $i \neq j$. where a_i is the interpreted as the intercept and b_i as the slope of a regression. The slope is also referred to as the sensitivity,or as factor loading, or risk exposure of asset i to the factor F.

Consider a portfolio of these n assets. Its returns are:

$$\widetilde{r}_p = \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} w_i b_i \widetilde{F} + \sum_{i=1}^{n} w_i \widetilde{\varepsilon}_i$$

Let us deøne: $a = \sum_{i=1}^{n} w_i a_i$, $b = \sum_{i=1}^{n} w_i b_i \widetilde{F}$, and $\widetilde{\varepsilon} = \sum_{i=1}^{n} w_i \widetilde{\varepsilon}_i$, substitute above and we have, again,

$$\widetilde{r}_p = a + b\widetilde{F} + \widetilde{\varepsilon}$$

Assume, as before, that

$$E[\widetilde{\varepsilon}] = 0, \text{and } E[(\widetilde{F} - E[\widetilde{F}]), \widetilde{\varepsilon}_i] = 0$$

Then, the variance of the error term is:

$$\sigma^2(\widetilde{\varepsilon}) = E[\varepsilon^2] = E[(\sum_{i=1}^{n} w_i \widetilde{\varepsilon}_i)(\sum_{i=1}^{n} w_j \widetilde{\varepsilon}_j)] = E[\sum_{i=1}^{n} w_i^2 \widetilde{\varepsilon}_i^2] = \sum_{i=1}^{n} w_i^2 \sigma^2(\widetilde{\varepsilon}_i)$$

The third equality follows from the assumption $E[\widetilde{\varepsilon}_i, \widetilde{\varepsilon}_j] = 0$. In order to show the e□ects of diversiøcation we assume that for all i, $\sigma^2(\widetilde{\varepsilon}_i) = s^2$, and that the portfolio is equally weighted, i.e. $\sigma^2(\widetilde{\varepsilon}) = \frac{1}{n}s^2$.

The variance of the portfolio is:

$$\sigma^2(\widetilde{r}_p) = b^2\sigma^2(\widetilde{F}) + \sigma^2(\widetilde{\varepsilon}) = b^2\sigma^2(\widetilde{F}) + \frac{1}{n}s^2$$

As n approaches inønity, the last term approaches zero, while the term $b^2\sigma^2(\widetilde{F})$ remains as is. Thus, the variance of the portfolio never becomes zero; only part of it can be diversiøed away. The part that can be diversiøed away is the idiosyngratic risk of the individual assets. The part that cannot be diversiøed is the exposure to risk caused by the systematic factors F that operates in the market.

7.2 The APT and the CAPAM

We will continue with the simple APT model discussed above and we will show that the one-factor model is equivalent to the CAPM.

As before, all assets satisfy the relationship:

$$\widetilde{r}_k = a_k + b_k\widetilde{F},$$

$k = 1, 2, ..., n$. Consider only two of them, the ith, and the jth. Here, we omit $\widetilde{\varepsilon}_i$ and $\widetilde{\varepsilon}_j$; we suppose that they cancel out. Construct a portfolio with w weight in asset i and $1 - w$ in asset j. The expected return of the portfolio is:

$$\widetilde{r}_p = wa_i + (1-w)a_j + [wb_i + (1-w)b_j]\widetilde{F}$$

Choose a particular w:

$$w = \frac{b_j}{b_j - b_i}$$

and substitute above we obtain:

$$r_p = \frac{a_i b_j}{b_j - b_i} - \frac{a_j b_i}{b_j - b_i}.$$

Note that this choice makes the return of the portfolio risk-free. Therefore, if there is a risk-free asset, the portfolio and the risk-free asset must yield the same rate of return. Thus,

$$r_f = r_p.$$

By substituting r_f for r_p in the previous expression we obtain:

$$\frac{a_j - r_f}{b_j} = \frac{a_i - r_f}{b_i} \equiv c \text{ (constant)},$$

which must hold for all i, j. Therefore, for each asset i:

$$a_i = r_f + b_i c,$$

and its expected return becomes:

$$E[\widetilde{r}_i] = a_i + b_i E[\widetilde{F}] = r_f + b_i(c + E[\widetilde{F}]).$$

Consider this expression. Since there are only two assets, if $E[\widetilde{F}] \neq r_f$ the ørst equality implies that there exist a unique number α such that

$$E[\widetilde{r}_i] = \alpha r_f + (1 - \alpha)E[\widetilde{F}].$$

This means that $a_i = \alpha r_f$ and $b_i = (1 - \alpha)$. These values, and the second equality yield $a_i = (1 - b_i)r_f$ and, from the deønition of c, $c = -r_f$. By using these expressions and thinking of $E[\widetilde{F}]$ as the expected return of the market portfolio we obtain

$$E[\widetilde{r}_i] = a_i + b_i E[\widetilde{F}] = r_f + b_i(E[\widetilde{r}_m] - r_f).$$

which is the to CAPM.

7.3 Arbitrage Pricing Theory

In a given period there are n securities, $i = 1, 2, ..., n$, and K factors $k = 1, 2..., K$. It is assumed that the number of assets is much larger than the number of factors. The (random) return of the ith asset is:

$$\begin{aligned} \widetilde{r}_i &= E[\widetilde{r}_i] + b_{i1}\widetilde{F}_1 + ... + b_{ik}\widetilde{F}_k + \widetilde{\varepsilon}_i \\ &\equiv E[\widetilde{r}_i] + \widetilde{u}_i + \widetilde{\varepsilon}_i \end{aligned}$$

where $\widetilde{u}_i \equiv b_{i1}\widetilde{F}_1 + ... + b_{ik}\widetilde{F}_k$ is the systematic risk, and $\widetilde{\varepsilon}_i$ the unsystematic risk. The coe□cient b_{ik} is the sensitivity of the ith asset to the kth factor, and $\widetilde{F}_k$ is the mean zero kth factor which a□ects the return of all assets.

Assumptions We assume that markets are perfectly competitive and that individuals have homogeneous beliefs. We also assume, as before, that $E[\widetilde{\varepsilon}_i, \widetilde{\varepsilon}_k] = 0$, and $E[\widetilde{\varepsilon}_i, \widetilde{F}_k] = 0$, for all $i \neq k$ and all time periods. Thus, unsystematic risks are uncorrelated across time, and securities and factors are independent of unsystematic risks.

A portfolio selected from the n assets which satisøes the following two conditions

(i) it requires no wealth and,

(ii) has no risk,

is called an arbitrage portfolio. Let such a portfolioo consist of n assets and its returns be

$$\widetilde{r}_p \equiv \Sigma w_i \widetilde{r}_i = \Sigma w_i E[\widetilde{r}_i] + \Sigma w_i b_{ik} \widetilde{F}_1 + ... + \Sigma w_i b_{iK} \widetilde{F}_K + \Sigma w_i \widetilde{\varepsilon}_i.$$

To satisfy the ørst condition the weights of such portfolio add-up to zero; i.e. $\Sigma w_i = 0$; with no wealth spent, the individual must use the proceeds from some assets to buy others. For the same reason, such portfolio must not yield any retrurn; if it did one could become very wealthy by paying nothing to aquire it.

Next, consider the second condition. In order for the portfolio to have no risk, the following are required:

(i) there must be many assets in the portfolio.

(ii) the weights of the assets must be small, and

(iii) the weights must be chosen so that the weighted average of the systematic risk of each factor is zero

The ørst two requirements are intented to eliminate the unsystematic risk. They are satisøed if n is large and that the weight of each asset is approximately equal to $\frac{1}{n}$. To satisfy the third condition, $w_i's$ must be chosen so as

$$\Sigma_i w_i b_{ij} = 0, \; k = 1, 2, ..., K$$

i.e. the weighted sum of the systematic risk component for each factor is equal to zero. In this way, all uncertainty is eliminated.

As a consequence of having no risk and no return, the expected returns of the arbitrage portfolio must be:

$$r_p = \Sigma w_i E[\widetilde{r}_i] = 0.$$

An algebraic consequence of the above statements (omitting the proof at this stage) is that there exists $K + 1$ coe□ cients $\lambda_0, \lambda_1, ..., \lambda_k$, such that

$$E[\widetilde{r}_i] = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + ... + \lambda_K b_{iK}.$$

The constants $\lambda_1, ..., \lambda_K$, are interpreted as the prices of the factors, and λ_0 as the risk-free rate.

To clarify the role of $\lambda's$ suppose that the ith asset has zero sensitivity to all factors except the kth. This is similar to the one-factor case discussed in the previous section. The expected return of the asset is:

$$E[\widetilde{r}_i] = \lambda_0 + b_{ik}\lambda_k = r_f + b_{ik}[E[\widetilde{r}_k] - r_f],$$

or,

$$\lambda_0 = r_f, \text{ and } \lambda_k = \frac{E[\widetilde{r}_k] - r_f}{b_{ik}}.$$

Next, suppose, in addition, that $b_{ik} = 1$ and substitute for b_{ik} above:

$$\lambda_k = E[\widetilde{r}_k] - r_f \equiv \delta_k - r_f$$

It follows that λ_0 is equal to the risk-free rate of return, λ_k is the risk premium of the kth factor, and δ_k is the expected return of the when the asset has zero sensitivity to all other factors, and unit sensitivity to the kth factor.

In general the APT model can be written as follows:

$$E[\widetilde{r}_i] = r_f + (\delta_1 - r_f)b_{1i} + (\delta_2 - r_f)b_{i2} + ... + (\delta_k - r_f)b_{ik}.$$

$i = 1, 2, .., n$. Here, as in the special case of the previous papagraph, δ_k is the expected return of a portfolio with unit sensitivity to the kth factor and zero sensitivity to all other factors,and λ_k is the risk premium of the kth factor, i.e.

$$\lambda_k = \delta_k - r_f, \text{ and } \lambda_o = r_f$$

$k = 1, 2, ..., K$.

The equations of the APT model can be estimated by linear regression, assuming that the vectors of return have a multinomial join distribution and the data on factors have been linearly transformed so that the transformed vectors are orthonormal.

Implementation Example: We continue to assume that there are n assets and K factors. Suppose that we have vectors with historical data on

(i) the expected returns of the assets (in percentage form) in the di□erent states of the world, and

(ii) the changes of the factors in each state (also in percentage form).

The next task is obtain linear transformation of the data on factors which is orthogonal; i.e. one in which the inner product of any two vectors is zero.

The risk premium of the jth factor, $k = 1, 2, ..., K$, factor can be estimated as the mean value of the transformed data on each factor ($\mu(\delta_k)$), and its price as

$$\lambda_k = \mu(\delta_k) - r_f.$$

The sensitivities of any asset, say the ith, can be computed in the same manner as the betas in the CAPM:

$$b_{ik} = \frac{Cov(\widetilde{r}_i, \widetilde{\delta}_k)}{Var(\widetilde{\delta}_k)}.$$

for all i and k. Then, the required return on asset i ,is

$$E[\widetilde{r}_i] = r_f + (\mu(\delta_1) - r_f)b_{1i} + (\mu(\delta_2) - r_f)b_{i2} + ... + (\mu(\delta_K) - r_f)b_{iK}$$

$i = 1, 2, .., n$. The above procedure can be used to detect whether there are arbitrage opportunities among the assets. Below we demonstrate such a case.

Suppose that there are only three assets and two factors,and that the portfolio of an investor consists of one third of each asset. By using historical data we have estimated the following mean of returns of assets $\mu(r_i)$, sensitivities of assets b_{ik}, and risk-premia of the factors $\mu(\delta_k)$. In addition, we have determined the required rate of return of the assets in equilibrium ($E[r_i]$). These estimates are presented in the following table:

Asset	$\mu(r_i)$	b_{i1}	b_{i2}	$E[r_i]$
x_1	11	.5	2	11
x_2	25	1	1.5	17
x_3	23	1.5	1	23

and

$$\mu(\delta_1) = 20, \ \mu(\delta_2) = 8$$

It is clear that an arbitrage opportunity exists for asset x_2 because the historical average rate of return is $17 whereas its required rate is 25. Thus, by reshuØing his/her portfolio the investor can realize some excess returns.

Change Portfolios To do so we form a riskless arbirtrage position. Let w_1, w_2, w_3 be the weights of such an arbitrage portfolio. As explained above, such a portfolio must satisfy the following conditions.

$$\begin{aligned} w_1 + w_2 + w_3 &= 0 \\ 0.5w_1 + 0.1w_2 + 1.5w_3 &= 0 \\ 2w_1 + 1.5w_1 + 1w_3 &= 0 \end{aligned}$$

Clearly, there are inønite solutions for $w_1, w_2,$ and w_3. because there are three unknowns, three equations, and each equation is equal to zero. Thus, we can øx arbitrarily the weight of one asset. One way is to put the maximum investment in asset x_2 since, as we have seen, the arbitrage opportunity exists for this asset. This means that the weight of x_2 is $w_2 = 2/3$ $(= 1 - 1/3)$ and, the above system of equations gives

$$w_1 = -1/3, \; w_2 = 2/3, \; w_3 = -1/3.$$

These are the weights of the riskless arbitrage portfolio. It is important to notice that the systematic risk of the new portfolio is the same as before: 1.0 for factor one,and 1.5 for factor 2. .The return, however, has changed; now it is 25%, whereas previously was only 19.67%.

8 Valuation of Securities

8.1 Equilibrium Valuation

8.1.1 Competitive Equilibrium with Market Securities

Consider an economy with:
·one consumption commodity,
·I individuals, $i = 1, 2, ..., I,$
·one future period with S possible states of the world, $s = 1, 2, ..., S$,and
· N market securities, $j = 1, 2, ..., N.$
The jth security yields a payo x_{js} in the sth future state. Let $p = (p_1, p_2, ..., p_N)$ be todays security prices, and $x_j = (x_{j1}, x_{j2}, ..., x_{jS})$ the payos of the jth security. Security prices and payos are denoted in units of the consumption commodity whose price is assumed to be one. Payos may be shown in matrix form as follows:

$$\mathbf{X} \equiv \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1S} \\ x_{21} & x_{22} & \dots & x_{2S} \\ \dots & \dots & \dots & \dots \\ x_{N1} & x_{N2} & \dots & x_{NS} \end{bmatrix}$$

Here a row represent the payos of asset j in the various states, whereas a columns represents payos the assets in state s. For example, x_{11} represents the payo of asset 1 in state 1. Whereas x_{NS} represents the payo of asset N in state S.

The ith individual has as initial endowments c_0^i units of the consumption commodity and $\overline{\mathbf{a}}_i = (\overline{a}_1^i, \overline{a}_2^i, ..., \overline{a}_N^i)$, $\overline{a}_j^i \geq 0$, units of securities. The goal of the individual is to choose todays consumption c_0^i and a portfolio of shares $a_i = (a_1^i, a_2^i, ..., a_N^i)$ so as to optimize his/her consumption preferences over time.

Let $u_0^i(c_0^i)$, $u_s^i(c_{1s}^i)$ be the functions which represent the consumption preferences of the individual in the present period and in the sth future state. They are assumed to be concave and monotone increasing; hence the individual consumes the entire payoff of his/her portfolio in each state; i.e. $c_{1s}^i = \sum_{j=1}^{N} a_j^i x_{js}$.

The theory of Expected Utility asserts that the in order to satisfy his/her goal the individual behaves as if he/she solves the following maximization problem:

$$\underset{\{c_0^i, a_1^i, a_2^i, \dots, a_N^i\}}{\text{maximize}} \quad u_0^i(c_0^i) + \sum_{s=1}^{S} \pi_s^i u_s^i\left(\sum_{j=1}^{N} a_j^i x_{js}\right)$$

$$\text{subject to } \; c_0^i + \sum_{j=1}^{N} a_j^i p_j = \overline{c}_0^i + \sum_{j=1}^{N} \overline{a}_j^i p_j$$

where $\pi^i = (\pi_1^i, \pi_2^i, \dots \pi_S^i)$, $\pi_s^i > 0$, are the subjective probabilities of each state.

Given the initial endowments and consumption preferences of all individuals, a vector of security prices supports the equilibrium if each and every individual takes these prices as given and by trading is able to trade satisfy his/her consumption preferences. This means several things:

·First, initial endowments dictate how much one can consume in the present and in any future state (i.e. the individual acts within the constraints of his/her budget).

·Second, aggregate consumption in the present and in any future state are equal to the supply of the consumption commodity , i.e.

$$\sum_{i=1}^{I} c_0^i = \sum_{i=1}^{I} \overline{c}_0^i \;, \text{ and} \sum_{i=1}^{I}\sum_{j=1}^{N} a_j^i x_{js} = \sum_{i=1}^{I}\sum_{j=1}^{N} \overline{a}_j^i x_{js}.$$

·Third, the market of each security clears; i.e.

$$\sum_{i=1}^{I} \overline{a}_j^i = \sum_{i=1}^{I} a_j^i, \quad \text{for } j = 1, 2, \dots, N.$$

To solve the above maximization problem we form the Lagrangian expression and after differentiating it with respect the control variables $(c_0^i, a_1^i, a_2^i, \dots, a_N^i)$, we obtain the first order conditions:

$$u_0^{i\prime}(c_0^i) = \mu$$

$$\sum_{s=1}^{S} \pi_s^i u_s^{i\prime}(c_{1s}^i) x_{js} = \mu p_j, \quad j = 1, 2, 3, \dots, N,$$

where μ is th Lagrangian multiplier, and the prime($'$) indicates a ørst order derivative.

In long hand, the above expression is a system of linear equations:

$$
\begin{aligned}
u_0^{i'} &= \mu \\
\pi_1^i u_1^{i'} x_{11} + \pi_2^i u_2^{i'} x_{12} + \ldots + \pi_s^i u_S^{i'} x_{1S} &= \mu p_1 \\
\pi_1^i u_1^{i'} x_{21} + \pi_2^i u_2^{i'} x_{22} + \ldots + \pi_s^i u_S^{i'} x_{2S} &= \mu p_2 \\
\pi_1^i u_1^{i'} x_{N1} + \pi_2^i u_2^{i'} x_{N2} + \ldots + \pi_s^i u_S^{i'} x_{NS} &= \mu p_N .
\end{aligned}
$$

(Here we have omitted the argument of the utility functions in order to simplify the presentation).

By dividing the above conditions with the ørst one we obtain:

$$\sum_{s=1}^{S} \pi_s^i \frac{u_s^{i'}(c_{1s}^i)}{u_0^{i'}(c_0^i)} x_{js} = p_j, \quad j = 1, 2, 3, \ldots.., N, \quad i = 1, 2, \ldots, I.$$

or, if we deøne

$$m_s^i \equiv \frac{u_s^{i'}(c_{1s}^i)}{u_0^{i'}(c_0^i)}, \text{ and } \phi_s \equiv \pi_s^i m_s^i \equiv \pi_s^i \frac{u_s^{i'}(c_{1s}^i)}{u_0^{i'}(c_0^i)}, \quad s = 1, 2, \ldots, S$$

and substitute in the previous expression we obtain:

$$p_j = \sum_{s=1}^{S} \pi_s^i m_s^i x_{js} = \sum_{s=1}^{S} \phi_s x_{js}, \quad j = 1, 2, 3, \ldots.., N, \quad i = 1, 2, \ldots, I.$$

In the last expression m_s^i is the MRS of the individual between present consumption, and consumption in the sth state. From the ørst equality, the price of a security is the sum of its expected payo□s, each one weighted by the individual□s MRS. The second equality makes the same price equal to the sum of its future payo□s weighted by some undeøned as yet quantities the ϕ_s□s. As we will see in a later section, these (called □state prices□) are the prices of artiøcial securities referred to as **contigent claims** and play an important role in the valuation of securities through arbitrage. Note that $\phi_s > 0$ from the assumption that $\pi_s^i > 0$ for all s and the fact that utilities are monotone increasing.

The former expression put in matrix notation becomes:

$$
\begin{bmatrix}
x_{11} & x_{12} & \ldots & x_{1S} \\
x_{21} & x_{22} & \ldots & x_{2S} \\
\ldots & \ldots & \ldots & \ldots \\
x_{N1} & x_{N2} & \ldots & x_{NS}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_S
\end{bmatrix}
=
\begin{bmatrix}
p_1 \\
p_2 \\
p_N
\end{bmatrix}
$$

This is a system of N equations can be solved for the S unknowns: $(\pi_s^i \frac{u_s^{i\prime}(c_{1s}^i)}{u_0^{i\prime}(c_0^i)} \equiv \phi_s$, $s = 1, 2, ..., S$). In matrix form they are:

$$\mathbf{X}\phi = \mathbf{p}.$$

From simple linear algebra the following is known:

· If $N = S$ and the returns of no asset can be duplicated by some combination of the returns of the remaining assets (i.e. if the returns of all assets are linearly independent), there is a unique portfolio which yields any particular pattern of consumption. For that reason the market is referred to as complete. Further, the matrix X is square and it can be inverted to give:

$$\phi = \mathbf{X}^{-1}\mathbf{p}.$$

Therefore, a competitive equilibrium is a necessary condition to derive a unique vector of state prices (to be explained below). Furthermore, if agents have homogeneous beliefs (i.e. they agree about the probability if each state), $\phi_s \equiv \pi_s^i m_s^i$ implies that for any two individuals, say the ith and the kth, $m_s^i = m_s^k$. Thus, the MRS~~s~~ of individuals are equal in equilibrium. Finally, if markets are complete, a competitive equilibrium is also a Preto optimum (and vise versa) as we will see in the sections which follow.

·If $N > S$, and the returns of at most S assets are ~~be~~ independent of each other and, ~~therefore,~~ there are many portfolios which can satisfy a particular pattern of consumption. This is so because any asset in such a portfolio can be replaced by a combination of other assets; so the portfolio is not unique. Furthermore, there **may not** be a unique vector of state prices. The latter relates to the question of the conditions which are required so that a portfolio of securities commands a unique price in the market. The so called □law of one price□ is an important property of any market economy. If the law is violated , there is possibility no arbitrage which will lead to the breakdown of the equilibrium, as we will see below. Finally, if

·$N < S$, not all consumption patterns can be generated with the existing assets. In this case there are not enough, so to speak, assets to satisfy all possible consumption preferences and the market is said to be incompete. In addition state prices cannot be determined. However, the valuation equations we derived above are still valid.

8.1.2 State Preference Theory. Arrow-Debreu Securities

We now introduce an important type of asset: the Arrow-Debreu Securities, also known as pure securities. These artiøcial assets generate claims which are contigent on the state which will occur; for that reason they are also referred to as state-contigent claims. Suppose that state-contigent claims are traded in a competitive market.

As a example, the sth pure security pays one unit of the consumption good in the sth state, and nothing in any other state; i.e. the payoff of the sth security is

$$(0,0,0,\ldots,1,0,0,0) \text{ where 1 is in the } sth \text{ position.}$$

Recall that all states have positive probability to occur, and let ϕ_s be the price of this security. It is obvious that $\phi_s > 0$, otherwise one would turn it into a “money pump” by obtaining free claims to the consumption good of state s, which must certainly be worth something in today's market.

If there exist S market securities the payoff of any market security can be duplicated with a portfolio of pure securities. For example, the jth market security which has payoff $(x_{j1},0,0,x_{j4},0,\ldots,x_{jS})$ is equivalent to a portfolio which consists of x_{j1} units of the first pure security, x_{j4} units of the fourth, and x_{jS} units of the Sth pure security.

Let $\mathbf{p} = (p_1,p_2,\ldots,p_N)$ be the prices of the N market securities which are also traded, and $\boldsymbol{\phi} = (\phi_1,\phi_2,\ldots,\phi_S)$ the prices of the pure securities (also known as **state prices**). The price of any market security, say the jth, must be equal to the price of a portfolio of pure securities which realizes the same pattern of payoffs across the states of nature; i.e.

$$p_j = \sum_s x_{js}\phi_s$$

This follows from the fact that both types of securities are traded in a competitive market. If equality does not hold in the above expression, there is a sure way to make a lot of money by repeatedly selling short the more expensive package and buying the cheaper one. To avoid problems like this we require the following condition to hold:

No Arbitrage condition : There is no arbitrage if

$$\mathbf{X}'\mathbf{a} \geqq 0 \text{ if and only if } \mathbf{p}'\mathbf{a} \geqq 0.$$

:In other words, if a portfolio yields non-negative payoffs in some state it must have a non-negative cost. The converse is also true: if a portfolio costs something to buy, it must yield something in some state of nature.

Above it was said that the payoff of any market security can be duplicated with a portfolio of pure securities. The convesre is also true: State contingent claims can be created by forming portfolios of market securities. For example, to find a state contingent claim: $(1,0,0,\ldots.,0)$ we solve the following problem:

$$(a_1,a_2,\ldots,a_n)'\mathbf{X} = (1,0,0,\ldots,0)'$$ check primes.

or

$$(a_1,a_2,\ldots,a_n) = (1,0,0,\ldots,0)^{T}\mathbf{X}^{-1}.$$

Competitive Equilibrium with Arrow-Debreu Securities We consider two periods (: present and future), I individuals, $i = 1, 2, 3, ...I$, S states of the world, and S pure securities, $s = 1, 2, 3, ...S$ (in other words we assume that there is a complete set of contigent claims). In this section I individuals, $i = 1, 2, 3, ...I$ the subscript of ith the individual is omitted, unless it becomes necessary to use it. An individual has initial endowments $(e_0, e_{11}, e_{12}, ...e_{1S})$ in units of the consumption good which he/she may consume or trade. It should be understood that future endowments are contigent commodities; e.g. endowment e_{js} consists of so many units of the consumption good which the individual will receive in the future only if the sth state occurs. The goal of the individual is to trade his/her initial endowments in order to achieve the most preferred consumption in the present and (whichever state attains in) the future. Let $(c_0, c_{11}, c_{12}, ...c_{1s})$ be a consumption bundle, where the ørst subscript refers to time, and the second to a state. The problem of the individual can be stated as a maximization problem:

$$\underset{\{c_0, c_{11}, c_{12}, ...c_{1s}\}}{\text{Maximize}} : u_0(c_0) + \sum_{s=1}^{S} \pi_s^i u_s^i(c_s)$$

subject to

$$c_0 + \sum_{s=1}^{S} \phi_s c_{1s} = e_0 + \sum_{s=1}^{S} \phi_s e_{1s}$$

where ϕ_s is the price of the sth pure security and π_s is the subjective probability of the ith individual for the sth state to occur. Note that the utility in period one may di□er from state to state. To proceed we form the Lagrangian expression

$$\begin{aligned} L &= u_0^i(c_0) + \pi_1^i u^i(c_{11}) + \pi_2^i u^i(c_{12}) + ... + \pi_3^i u^i(c_{1S}) \\ &\quad + \theta[(e_0 - c_0) + \phi_1(e_{11} - c_{11}) + \phi_2(e_{12} - c_{12}) + ... + \phi_S(e_{1S} - c_{1S})], \end{aligned}$$

where θ is the Lagrangian multiplier. From micro-economics we know that θ is the marginal utility of wealth of the individual.

The ørst order conditions of this problem are:

$$\pi_s^i u_s^{i\prime}(c_{1s}) = \theta \phi_s,$$

$s = 1, 2, 3, ...S,$

$$u_0^{i\prime}(c_0) = \theta,$$

plus the constraint. By taking the ratio of these expressions, we obtain

$$\pi_s^i \frac{u_s^{i\prime}(c_{1s})}{u_0^{i\prime}(c_0)} = \phi_s,$$

$s = 1, 2, 3, ...S$.

The ørst order conditions show that he marginal rate of substitution (MRS) between consumption in state s and consumption in period zero weighted by the probability of the the sth state is equal to the state price ϕ_s. Note that if all individuals have homogeneous expectations, the MRS for each state are the same for all individuals.

Pareto Optimality and Competitive Equilibrium According to Pareto a Social Plannerђs concern should be to allocate aggregate output among the I members of the community in an optimal way. An allocation is Pareto optimal if it is feasible and if it cannot increase the utility of one individual without decreasing the utility of another. In what follows it is assumed that individuals have homogeneous expectations.

We know from microeconomics that an allocation is Pareto Optimal if weights $\{\lambda_1, \lambda_2, ..., \lambda_I\}$ can be found, such that the following expression is maximized:

$$\sum_{i=1}^{I} \lambda_i \{u_0^{ii}(c_0^i) + \pi_1^i u_1^{ii}(c_{11}^i) + \pi_2 u_2^{ii}(c_{12}^i) + ... + \pi_S^i u_S^{ii}(c_{1S}^i)\}$$

subject to

$$\sum_{i=1}^{I} c_0^i = c_0 \text{ and } \sum_{i=1}^{I} c_{1s}^i = c_s, \; s = 1, 2, 3, ...S.$$

We form the Lagrangian:

$$\begin{aligned} L &= \sum_{i=1}^{I} \lambda_i \{ \; u_0^{ii}(c_0^i) + \pi_1^i u_1^{ii}(c_{11}^i) + \pi_2^i u_2^{ii}(c_{12}^i) + ... + \pi_S^i u_S^{ii}(c_{1S}^i)\} \\ &\quad + \phi_0 + \{c_0 - \sum_{i=1}^{I} c_0^i\} + \sum_{s=1}^{S} \phi_S (c_S - \sum_{i=1}^{I} c_{1s}^i) \end{aligned}$$

where $\phi_0, \phi_1, \phi_2, ..., \phi_S$ are the Lagrangian multipliers. The ørst order conditions are:

$$\lambda_i u_0^{i/}(c_0^i) = \phi_0, \; i = 1, 2, ..., I$$

$$\lambda_i \pi_s u_s^{i/}(c_{1s}^i) = \phi_s, \; s = 1, 2, ..., S, \; i = 1, 2, ..., I.$$

By taking ratios of these expressions we obtain:

$$\pi_s^i \frac{u_s^{i/}(c_{1s}^i)}{u_0^{i/}(c_0^i)} = \frac{\phi_s}{\phi_0}.$$

These are the same conditions we derived in the case of the competitive equilibrium if we let $\phi_0 = 1$ and $\lambda_i = 1/\theta_i$. Using these, the above expression becomes:

$$\phi_s = \pi_s^i \frac{u_s^{i\prime}(c_{1s}^i)}{u_0^{i\prime}(c_0^i)}, \quad s = 1, 2, ..., S, \; i = 1, 2, ..., I.$$

By comparing this expression with the one obtained under competitive equilibrium we conclude that, a competitive equilibrium implies Pareto optimum. The converse can be also shown to hold. However, in an incomplete market, a competitive equilibrium is not necessarily Pareto optimum.

Equilibrium in an Economy with Securities Markets From the Pareto optimality condition, we have deduced that in a complete market the vector of prices pure securities is equal to the vector of MRS of the ith individual weighted by the state probabilities, i.e. for $s = 1, 2, ..., S$

$$\phi_s = \pi_s^i \frac{u_s^{i\prime}}{u_0^{i\prime}}$$

or,

$$\mathbf{X}\phi = \mathbf{p}$$

where $\mathbf{X}$ is the matrix of payoffs, ϕ the vector of prices of pure securities, and $\mathbf{p}$ the vector of prices of market securities.

Since the matrix $\mathbf{X}$ is square and non-singular, it can be solved for the state prices. This is, again, the Fisher separation property.

8.2 Risk-Neutral Valuation

We say that we have complete markets if we can define prices for pure securities for each possible state of the world. ~~Below~~ we will show that in a economy with complete markets a competitive equilibrium is also a Pareto optimum. It has been shown above

Given the payoff matrix $\mathbf{X}$, the market price of N assets $\mathbf{p} = (p_1, p_2, ..., p_N)$, and a portfolio of assets $a = (a_1, a_2, ..., a_N)$, the possibility of arbitrage exists if either of the following two conditions hold:

$$\begin{aligned} &1. \;\; \mathbf{p}'\mathbf{a} \;\leqq\; 0 \;\; \text{imlpies } \mathbf{X}'\mathbf{a} > 0 \\ &2. \;\; \mathbf{p}'\mathbf{a} \;<\; 0 \;\; \text{implies } \mathbf{X}'\mathbf{a} \geqslant 0 \end{aligned}$$

Where: $\mathbf{p}'\mathbf{a} = \sum_{j=1}^{N} p_j a_j$ is the cost of the portfolio, and $\mathbf{X}'\mathbf{a}$ is its payoff. The absence of arbitrage guarantees that non-negative returns have non-negative costs.

Theorem. No arbitrage opportunities exist if and only if, there exists a $\phi > 0$ such that

$$\mathbf{p} = \mathbf{X}\phi$$

where $\phi = (\phi_1, \phi_2, ..., \phi_S)$.

In other words, the existence of prices for S pure securities is a necessary and su□ cient condition for the absence of arbitrage: If the non-arbitrage condition holds, there exists positive state-prices (i.e. prices if pure securities). Conversly, if there exist state prices the corresponding security prices do not admit arbitrage. Note that this condition is the expression we derived at the end of the previous section. It asserts that no arbitrage opportunities exist if we can ønd a vector of state prices. Similarly, if we have a vector of state prices we can be certain that there are no arbitrage opportunities.

8.2.1 Asset Pricing and the No Arbitrage Condition

[We start by warning that in what follows there is a change of notation in order to be in line with the literature on the subject; the letter $\mathbf{S}$ will be used for the vector of prices of market securities instead of $\mathbf{p}$ used so far]

It is assumed that in an economy the prices of market securities do not admit arbitrage. There are $N+1$ assets, risk-free asset and N risky ones. The $0th$ asset is a bond which pays $1+r$ in each state of the world. Its market price in the current period must be equal to one i.e. $S_0 = 1$; otherwise there would be arbitrage. Divide all the entries of the payo□ matrix $\mathbf{X}$ by $1+r$ to obtain the **discounted values of the payoffs** $\widehat{\mathbf{X}}$:

$$\widehat{\mathbf{X}} = \begin{bmatrix} 1 & 1 & ... & 1 \\ \widehat{x}_{11} & \widehat{x}_{12} & ... & \widehat{x}_{1S} \\ ... & ... & ... & ... \\ \widehat{x}_{N1} & \widehat{x}_{N2} & ... & \widehat{x}_{NS} \end{bmatrix}$$

where

$$\widehat{x}_{js} = \frac{x_{js}}{1+r},$$

From the assumption of no arbitrage, we know that there exist state prices which satisfy the equations:

$$\begin{aligned} \mathbf{S} &= \mathbf{X}\phi \\ &\equiv \widehat{\mathbf{X}}(1+r)\phi \equiv \widehat{\mathbf{X}}\widehat{\pi}, \end{aligned}$$

where

$$\widehat{\pi}_s = (1+r)\phi_s,\ s = 1, 2, ..., S.$$

The ørst equation is

$$\sum_{s=0}^{N} \phi_s = \frac{1}{1+r}$$

$$\sum_{s=1}^{S} \widehat{\pi}_s = \sum_{s=1}^{S} \widehat{\pi}_s (1+r)\phi_s \equiv 1.$$

?

The former expression a□rms that the sum of state prices is equal to the discount value of riskless borrowing. The last one asserts that the □modiøed□ state prices $(1+r)\phi_s$ add-up to one. This property and the fact that ϕ_s is greater than zero for all s, as the theorem requires, allows one to treat these □modiøed□ state prices as probabilities. We call the set of $\widehat{\pi}_s$□s **risk-adjusted (risk-neutral)** probabilities, and the vector $\widehat{\boldsymbol{\pi}}$ a **risk neutral probability measure.** Note that the risk-neutral probabilities are not the same as the □true□ probabilities. However, each $\widehat{\pi}_s$ is indirectly related to the □true□ probability of the sth state π_s because in a competitive equilibrium $\phi_s = \pi_s \frac{u_s^i}{u_0^{i'}}$ for all $s = 1, 2, ..., S$. (note: no homog. expect. are required)

Using the above expressions, we write the price of the jth security as:

$$S_j = \sum_{s=1}^{S} \phi_s (1+r) \frac{x_{js}}{1+r} = \sum_{s=1}^{S} \widehat{\pi}_s \widehat{x}_{js},\ \ j = 1, 2, 3, ..., N,$$

i.e. S_j is the expected price of its modiøed payo□s where of the expectation is taken with respect to the risk-adjusted probabilities. This is a fundamental relation used in the pricing of securities through arbitrage.

Suppose a two period ønancial market is arbitrage free. This implies two equivalent conditions:
(i) there exist a unique state price vector, and
(ii) there exist a risk-neutral probability measure.
The existence of a unique state price vector is guaranteed if there exist S securities with linearly independent payo□s and security prices are determined in a competitive equilibrium. Then, state prices can be determined, as we have seen before, which can be used to derive a risk-neutral probability measure. If there are more securities than S (i,e, $N > S$) their prices can be determined by the arbitrage. In practice, we assume that these conditions hold. However, it is not clear that there are more securities than states of the world, in which case the market may not be complete. The existence of non-marketable securities, (side-bets) such as private insurance contracts, is an indication that indeed the market may not be complete.

Digression. To see the relation of discounted values to the state prices, suppose we have a only two states, one risk-free and one risky asset. In a two state world, the payo□ matrix is:

$$\mathbf{X} = \begin{bmatrix} 1+r & 1+r \\ x_{11} & x_{12} \end{bmatrix}$$

After dividing the entries of $\mathbf{X}$ by $1+r$, the expression

$$\mathbf{S} = \mathbf{X}\phi$$

becomes

$$\begin{bmatrix} 1 \\ S_1 \end{bmatrix} = (1+r) \begin{bmatrix} 1 & 1 \\ \widehat{x}_{11} & \widehat{x}_{12} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

which gives

$$\frac{1}{1+r} = (\phi_1 + \phi_2)$$

and

$$\begin{aligned} S_1 &= \frac{1}{1+r}[\phi_1(1+r)x_{11} + \phi_2(1+r)x_{12}] \equiv \frac{1}{1+r}[x_{11}\widehat{\pi}_1 + x_{12}\widehat{\pi}_2] \\ &= \widehat{x}_{11}\widehat{\pi}_1 + \widehat{x}_{12}\widehat{\pi}_2 \end{aligned}$$

where

$$\widehat{\pi}_s \equiv \phi_s(1+r), \text{ and } \widehat{x}_{js} = \frac{x_{js}}{1+r}, \;\; j = 1, 2.$$

The last expression combined with $\frac{1}{1+r} = (\phi_1 + \phi_2)$ imply

$$\phi_1(1+r) + \phi_2(1+r) \equiv \widehat{\pi}_1 + \widehat{\pi}_2 = 1$$

which explains why $\widehat{\pi}_1$and $\widehat{\pi}_2$ can be interpreted as probabilities.

In general with one risk-free asset and $N = S$, risky securities we have:

$$\sum_{s=1}^{S} \widehat{\pi}_j = \sum_{s=1}^{S} \phi_s(1+r) = 1,$$

Example. The undiscounted payo□ matrix is

$$\mathbf{X} = \begin{bmatrix} 1.1 & 1.1 \\ 150 & 100 \end{bmatrix}.$$

The equations $\mathbf{S} = \mathbf{X}\phi$ become

$$\begin{bmatrix} 1 \\ 100 \end{bmatrix} = \begin{bmatrix} 1.1 & 1.1 \\ 150 & 100 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.$$

Solving this system of equations gives:

$$\begin{aligned}\phi_1 &= 0.1818\\ \phi_2 &= 0.7273\end{aligned}$$

Then, from the risk-adjusted probabilities: $\widetilde{\pi}_s = \phi_s(1+r)$ we obtain:

$$\widehat{\pi}_1 = 1.1 * 0.1818 = 0.2$$

$$\widehat{\pi}_2 = 1.1 * 0.7273 = 0.8.$$

Any other assets may be priced using these values. For example, if another security is introduced in this market which has discounted payo□s (120,60), respectively in each of the two states, its price in period zero should be:

$$120 * 0.2 + 60 * 0.8 = 72.$$

If the payo□s are undiscounted itsprice should be:

$$120 * 0.1818 + 60 * 0.7273 = 65.454.$$

9 Multiperiod Model

9.1 Summary:

Deøne

·information Structure,
·dividend processes,
·trading strategies,
·admissible trading strategies,
·consumption plans adapted to information,
·marketed consumption plans,
·no arbitrage (it does not admit the possibility of something created from nothing).

·Create a discounted price system and accumulated dividend process. Such a system exists if and only if there exists a martingale measure which admits no arbitrage opportunities.

·Derive the Martingale measure from the prices of securities which correspond to the information structure (as shown by the information tree). Work backwards by deriving conditional probabilities for each branch of the tree. The martingale measure is the unconditional probabilities implied by the conditional probabilities.

·The Martingale property (as a necessary condition) allows us to compute the prices of any security over time.

·Long-lived securities (or derivative securities) are equivalent to marketed consumption plans. Marketed consumption plans have a unique cum-dividend price at any t, $t = 0, 1, 2, ..., T$. The prices of any consumption plan can be computed by evaluating a conditional expectation under the martingale measure.

9.2 Equilibrium Valuation

9.2.1 Information Structure

Suppoes that time is divided into time periods, say, from period zero to T. A **state of nature** is a complete description of the exogenous economic environment between period zero and T. At time zero individuals expect that some state from a given collection of states will prevail in T, but they are uncertain as to which one. As time progresses, information becomes available which narrows down the possibilities, until period T when the true state is revealed.

Denote by ω a state of nature between period zero and T and by Ω the collection of states which are candidates to realize. The unfolding of the information between period zero and T, is referred to as the **information structure** and best represented by an **event tree**

As any tree, the event tree has a trunk, nodes and branches. The trunk supports the whole tree, and corresponds to the information structure available at time zero, when all states are indistinguishable. Nodes correspond to the information which will be revealed in some period prior to T. A branch that emanates from a node constitutes an **event** and it is a collection of states. The collection of events at time t is called a **partition** of Ω and denoted by F_t. When all uncertainty is revealed at time T the economy will be at some terminal branch of the event tree.

Clearly $F_0 = \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$.

In each period t some event out of the several possible ones will realize. Given that the intersection of events is empty, this means that only the states included in that event may occur in the future. The remaining events (and states) will be cut-o□ from the information tree. This □thinning□over time of the event tree is a consequence of the gradual resolution of uncertainty. Thus, a partition F_t is øner than another one F_s if $s > t$.

In the above event tree there are øve possible paths:
(a_0, a_{11}, ω_1),(a_0, a_{11}, ω_2),(a_0, a_{11}, ω_3),(a_0, a_{12}, ω_4),(a_0, a_{12}, ω_5).

The intersection of these paths is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$. At time zero the individual cannot distinguish between any of the øve paths that the economic environment may take; thus, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ He/she only knows that in period one the economy will be either in event a_{11} or in event a_{12}. If a_{11} occurs only states $\{\omega_1, \omega_2, \omega_3\}$ are possible from then on, while states $\{\omega_4, \omega_5\}$, will never occur. If on the other hand a_{12} occurs, the future state will be either ω_4 or ω_5

while $\{\omega_1, \omega_2, \omega_3\}$ will never realize. The partition F_1 and F_2 are, respectively, $F_1\{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}\}$ and $F_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}\}$ In what follows we will omit the subscript s referring to a state in an event and use a_t for any event in period t.

9.3 Competitive equilibrium with Arrow-Debreu Securities

In an economy there exist:
·one consumption commodity,
·I individuals, $i = 1, 2, ..., I$,
·T periods, $t = 0, 1, 2, ..., T$,
·S states of nature, $s = 1, 2, ..., S$. The information structure $F = \{F_t;\ t = 0, 1, 2, ..., T\ \}$ is assumed to be known to all individuals. The probability of an event in period t can be calculated using Bayes rule:

$$\pi_{a_t} = \sum_{\omega \in \alpha_t} \pi_\omega$$

·A number of contigent claims: $\{e_0^i, e_{a_t}^i, a_t \in F_t; t = 1, 2, ..., T\}$. In a multiperiod model a contigent claim pays one unit of the consumption commodity at a particular time and only if a particular event occurs, and nothing in all other cases; i.e. it pays at some date $t > 1$, in some event $a_t \in F_t$, and nothing otherwise. Suppose that each of the N individuals is endowed with an initial quantity of contigent claims. The individual plans to optimize his/her consumption preferences by trading in in contigent claims Thus, as before he/she solves the following problem:

$$\text{maximize} \quad u_0^i(c_0^i) + \sum_{t=1}^{T} \sum_{\alpha_t \in F_t} \pi_{a_t} u_t^i(c_{\alpha_t}^i)$$

with respect to

$$\{c_0^i, c_{\alpha_t}^i; \alpha_t \in F_t, t = 0, 1, 2, ..., T\}$$

subject to his intertemporal budget constraint:

$$\phi_0 c_0^i + \sum_{t=1}^{T} \sum_{\alpha_t \in F_t} \phi_{a_t} c_{a_t}^i = \phi_0 e_0^i + \sum_{t=1}^{T} \sum_{\alpha_t \in F_t} \phi_{a_t} e_{a_t}^i.$$

The ørst order conditions of the Lagrangian optimization problem are:

$$\pi_{a_t} \frac{\partial u_t^i(c_{\alpha_t}^i)}{\partial c_{\alpha_t}^i} = \gamma_i \phi_{\alpha_t},$$

$$\frac{\partial u_0^i(c_{\alpha_0})}{\partial c_0} = \gamma_i \phi_0,$$

for all events in F_t and all t. Here γ_i is the Lagrangian multiplier. By taking the ratio of these expressions, we obtain

$$\frac{\phi_{\alpha_t}}{\phi_0} \equiv \frac{\pi_{a_t} u_t^{i\prime}(c_{\alpha_t})}{u_0^{i\prime}(c_{\alpha_0})}$$

for all α_t in F_t and all t.

In a manner analogous to the two-period model the optimization problem of Pareto's social planner is to solve the following maximization problem for a set of positive weights $\{\lambda_i\}$, $i = 1, 2, ..., I$,

$$\text{maximize} \sum_{i=1}^{I} (\lambda_i(u_0^i(c_0^i) + \sum_{t=1}^{T} \sum_{\alpha_t \in F_t} \pi_{a_t} u_t^i(c_{\alpha_t}^i))$$

with respect to

$$\{c_0^i, c_{\alpha_t}^i; \alpha_t \in F_t, \ i = 1, 2, ..., I, \ t = 0, 1, 2, ..., T\}$$

subject to

$$\sum_{i=1}^{I} c_0^i = \sum_{i=1}^{I} e_0^i, \text{ and } \sum_{i=1}^{I} c_{\alpha_t}^i = \sum_{i=1}^{I} e_{\alpha_t}^i$$

for all i all α_t in F_t and all t. The first order conditions of the Lagrangian optimization problem are:

$$\lambda_i \pi_{a_t} \frac{\partial u_t^i(c_{\alpha_t}^i)}{\partial c_{\alpha_t}^i} = \phi_{\alpha_t},$$

$$\lambda_i \frac{\partial u_0^i(c_{\alpha_0}^i)}{\partial c_0} = \phi_0,$$

for all i, all events in F_t, and all t. Taking the ratio of these expressions, we obtain

$$\frac{\phi_{\alpha_t}}{\phi_0} \equiv \frac{\pi_{a_t} u_t^{i\prime}(c_{\alpha_t})}{u_0^{i\prime}(c_{\alpha_0})}$$

for i all α_t in F_t and all t.

The solutions to the Pareto problem are the same as those of the competitive equilibrium if we let $\lambda_i = 1/\gamma_i$. (What does this say about λ_i's and "social justice"?) Again a competitive equilibrium is Pareto optimum if there exists a complete set of contigent claims; i.e. there exist claims for all α_t in F_t and all t.

Under the above conditions Individuals make their plans at time zero and execute them during the rest of the time. The market need operate only at time zero and may not re-open in any future period. (This is much like the Walrasian markets where there is a complete set of prices for all present and future commodities.) The requirement for this result is that individuals have rational expectations.

Rational Expectations Consider an event α_s at time t on the condition that event α_t, has occurred, $s \geq t$. The probability of α_s is denoted by $\pi_{\alpha_s}(\alpha_t)$, and it numerical value is, by Bayes rule,

$$\begin{aligned} \pi_{\alpha_s}(\alpha_t) &= \frac{\pi_{\alpha_s}}{\pi_{\alpha_t}} \text{ if } \alpha_s \subseteq \alpha_t, \text{ and} \\ &= 0 \text{ if } \alpha_s \nsubseteq \alpha_t \end{aligned}$$

Next, consider the prices of contigent claim at a time prior to their payoff day. Let

$$\phi_{\alpha_s}(\alpha_t)$$

be the price of a contigent claim at time t and event α_t paying o in event α_s, $s \geq t$. We say that individuals have rational expectations if they evaluate probabilities correctly and if they believe that prices of these contigent claims evolve as according to the following rule. .

$$\begin{aligned} \phi_{\alpha_s}(\alpha_t) &= \frac{\phi_{\alpha_s}}{\phi_{\alpha_t}} \text{ if } \alpha_s \subseteq \alpha_t, \text{ and} \\ &= 0 \text{ if } \alpha_s \nsubseteq \alpha_t \end{aligned}$$

Suppose that there is a complete set of contigent claims, individuals have rational expectations, that markets operate in every period. It can be shown that under these conditions (i.e. in a **competitive rational expectations equilibrium**) the optimizing decisions of the agents will not deviate from those made at time zero; they will just repeat over time the decisions made when the market operates only once.

When there does not exist a complete set of Arrow -Debreu securities, not all consumption patterns may be satisøed by trading in contigent claims alone. Then, it may be necessary that markets remain open in periods other than zero. This is so because under certain conditions individuals may complete the market and obtain any desired pattern of consumption by trading in market securities.

9.4 Competitive equilibrium with Market Securities

In this market there exist $N + 1$ securities that pay non-negative dividends in every period and in every event between time zero and time T. At the beginning of each period, the ex-dividend price of securities are known with certainty . Future prices are uncertain because it is not known which events in the future. In general, we assume that a security has a positive ex-dividend price at time t if there is a positive probability that it will yield dividends in some future event.

Notation A **stochastic process** is a collection of random variables indexed by time. If the value of a random variable referring to time t does not vary across

all states in an event $\alpha_t \in F_t$, is said that the random variable is **measurable** with respect to F_t. Processes for which the random variables are measurable for all $t's$ are said to be **adapted** to F (which is the collection of all F_t's).

In a multiperiod model individuals deal with stochastic processes, such as the prices and the dividends of market securities. Their decisions are themselves processes, such as consumption plans and trading strategies. It is natural to require that such processes are adapted to the constraints of the information structure as represented by the event tree. In what follows we will assume that all processes are adapted to the information structure F..

Notation

$\boldsymbol{\theta}^i = (\bar{\theta}_1^i, \bar{\theta}_2^i, ..., \bar{\theta}_N^i)$: the vector of initial endowments of the ith individual in market securities.

$c(t) = \{c(a_t), \alpha_t \in F_t\}$ a consumption plan of period t and event a_t.

$c = \{c(t), t = 0, 1, ..., T\}$:a consupmtion process adapted to F.

$\boldsymbol{\theta}_j(t) = \{\theta_j(a_t), \alpha_t \in F_t\}; j = 1, 2, ..., N\}$: the trading plan for security j in period t and event a_t.

$\boldsymbol{\theta} = \{\boldsymbol{\theta}_j(t),\ j = 0, 1, 2, ..., N,\ t = 1, 2, ..., T\}$: a trading process adapted to F.

$S_j = \{S_j(t),\ t = 1, 2, ..., T\}$: the price process of the jth security.

$\mathbf{S}(t) = (S_0(t), S_2(t), ..., S_N(t))$, the vector of security prices in period t.

$x_j = \{x_j(t),\ t = 0, 1, ..., T\}$: the dividend process of the jth security adapted to F.

$\mathbf{X}(t) = (x_0(t), x_2(t), ..., x_N(t))$, the vector of dividends of period t.

9.5 The individual

At time zero, the ith individual has an initial endowment in the N market securities $\boldsymbol{\theta}^i = (\bar{\theta}_1^i, \bar{\theta}_2^i, ..., \bar{\theta}_N^i)$ which plans to trade in order to attain a consumption plan which maximizes his/her consumption preferences over time. This requires that he/she chooses in advance what to consume in all periods and across all events which may occur Such a consumption plan can be implemented only by trading in securities. Thus, the individual must devise a trading strategy which will allow him/her to finance the optimum consumption plan in the following sense: The trading strategy should be a complete description of the purchase or sale of each of the N securities in each of the T periods. The net proceeds from the security transactions in each period will used to finance the consumption of the period. Such trading policies are called **admissible** policies. Making such a plan is indeed a complicated task, but we can brake it down into smaller period-by-period decisions which are easier to understand and, then, connect them into an overall plan. Formally this is done through **Dynamic Programming**; here we will describe its logic and give an example.

Place yourself at the beginning of a period t. The consumer knows the event which has realized for the period and, thus, which states may possibly occur in the future. In addition, he/she has a number of securities which were purchased in period

$t-1$ and knows the prices the period; thus he/she knows the value of wealth at that point of time is known. The goal is (a) to consume an optimum amount for the period, and (b) to buy another portfolio for period t+1. This portfolio should allow the consumer to maximize consumption preferences in whichever event might occur, and save enough to buy another portfolio for period t+2. The decision processes of all periods are tied together by requiring that at time zero the consumer should be able to a□ord the portfolios he/she plans to buy in each time period. In other words, the consumer should choose a trading strategy in securities which ønances a feasible plan which is maximizes overall consumption preferences.

But how does one know what to provide for future consumption when one is not sure which even will occur? The trick is to start thinking backwards in time. At time T all uncertainty is resolved. If the consumer knew which event will realize in $T-1$ (and, thus, the value of the starting portfolio) he/she would make optimizing decisions as in the two-period model described in previous chapters. Since he/she is uncertain about the true state, the individual makes optimal consumption and saving decisions **conditional** on the event that might occur at that time. Next, we can move one period earlier, in $T-2$, and proceed in an analogous manner. In each step, decisions are conditional on the information available up to that time (i.e. which events may occur during the period and, thus, which states may occur in the future). This procedure is repeated until the present period, where the state and the value of the initial endowment is known. With this information, the budgetary requirements of each period and each event are precisely determined by going forward in time and making deønite the conditional plans.

Suppose we are in period T and some state ω has realized. The individual has a portfolio of securities $\boldsymbol{\theta}(T)$ which were purchased in $T-1$. The value of the individual□s wealth is the dividends which the portfolio will yield at that state. Since this is the last period, the individual will consume all of his/her wealth, i.e.

$$c(T) = \boldsymbol{\theta}(T)'\mathbf{X}(T) = W(T)$$

and his/her utility will be $u_T(c(T))$. In period $T-1$, the starting portfolio is $\boldsymbol{\theta}(T-1)$, purchased in $T-2$, and the wealth of the individual is

$$W(T-1) = \boldsymbol{\theta}(T-1)'\mathbf{S}(T-1)+\mathbf{X}(T-1).$$

$W(T-1)$ will be allocated into consume for the period and into buying a portfolio of securities which will provide for the consumption of T. Thus, the wealth constraint for $T-1$ is

$$\boldsymbol{\theta}(T-1)'\mathbf{S}(T-1)+\mathbf{X}(T-1) = c(T-1) + \boldsymbol{\theta}(T)'\mathbf{S}(T-1)$$

In order to maximize his/her consumption preference the individual will solve the problem:

$$\max \quad u_T(c(T-1)) + E[u_T(c(T)) \mid F_{T-1}]$$

subject to the above constraint. Here $E[. \mid .]$ denotes the conditional expectation operator; utility is maximized conditional on the information structure of period $T-1$. Denote the maximum value of the above problem (i.e. the value after the consumption plan and optimal strategy is chosen) by:

$$V(W(T-1);F_{T-1})$$

It is easy to see that proceeding in the same manner for period $T-2$ the utility of $W(T-2)$, after the optimum consumption plan and trading strategy is chosen, is:

$$V(W(T-2);F_{T-2}) = \max \quad u_T(c(T-2)) + E[V(W(T-1);F_{T-1}) \mid F_{T-2}],$$

where the maximization has been made subject to

$$W(T-2) = \boldsymbol{\theta}(T-2)'\mathbf{S}(T-2)+\mathbf{X}(T-2) = c(T-2) + \boldsymbol{\theta}(T-1)'\mathbf{S}(T-2).$$

In general, in period t the utility of wealth is

$$V(W(t);F_t) = \max \quad u_t(c(t)) + E[V(W(t+1);F_{t+1}) \mid F_t]$$

where the maximization has been made subject to

$$W(t) = \boldsymbol{\theta}(t)'\mathbf{S}(t)+\mathbf{X}(t) = c(t) + \boldsymbol{\theta}(t+1)'\mathbf{S}(t).$$

It can be shown that $V(W(t);F_t)$, is increasing and strictly concave in $W(t)$. For $t > 0$ the value of wealth is a random variable conditional on the information up the that period. At $t = 0$ the value of wealth is known from the initial endowments of the individual. Given the optimizing decisions at $t = 0$ we can go forward in time and make concrete all the conditional plans.

By di□erentiating both sides of this expression for $W(t)$ we obtain:

$$u_t'(c(t)) = V_w(W(t);F_t)$$

where V_w is the partial derivative of V with respect to $W(t)$, thus along the optimal path the marginal utility of wealth must be equal to the marginal utility of consumption.

As in a two-period model, the price of a market security in period t must be equal to the price of a portfolio of contigent claims which realizes the same pattern of payo□s across present and future states. Thus, the price of the jth security at time t and event α_t can be express in terms of the prices of claims paying in a o□ at time

α_s, $s > t$ as follows:.

$$
\begin{aligned}
S_j(a_t, t) &= \sum_{s=t+1}^{T} \sum_{\substack{\alpha_s \in F_s \\ a_s \subseteq a_t}} \phi_{\alpha_s}(\alpha_t) x_j(\alpha_s) \\
&= \sum_{s=t+1}^{T} \sum_{\substack{\alpha_s \in F_s \\ a_s \subseteq a_t}} \frac{\pi_{a_s} u_s^{i\prime}(c^i_{\alpha_s})}{u_t^{i\prime}(c^i_{\alpha_t})} x_j(\alpha_s) \\
&= \sum_{\substack{\alpha_{t+1} \in F_{t+1} \\ a_{t+1} \subseteq a_t}} \frac{\pi_{a_{t+1}} u_{t+1}^{i\prime}(c^i_{\alpha_{t+1}})}{u_t^{i\prime}(c^i_{\alpha_t})} (x_j(\alpha_{t+1}) + S_j(a_{t+1}, t+1))
\end{aligned}
$$

The second expression above results from the optimization conditions we obtained above for the competitive equilibrium with state contigent claims. The third, resultes from the second if we split it in two terms; one corresponding to time $t+1$ and one to the rest of the periods. Then` using the **law of iterated expectations** we express dividends in terms of prices. The last expression can be written in an equivalent way using the expectation operator.

$$
\begin{aligned}
S_j(t) &= E\left[\sum_{s=t+1}^{T} \frac{\pi_{a_s} u_s^{i\prime}(c^i(t+1))}{u_t^{i\prime}(c^i(t))} x_j(s) \mid F_t \right] \\
&= E\left[\frac{u_{t+1}^{i\prime}(c^i(t+1))}{u_t^{i\prime}(c^i(t))} (x_j(t+1) + S_j(t+1)) \mid F_t \right]
\end{aligned}
$$

where now we terms are random variables. These equilibrium pricing formulas should be should be contrasted with those of the two-period model.

An Example of a Trading strategy In an economy there are øve states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, three periods $t = 0, 1, 2$, and three market securities. The securities pay dividends only in the last period $t = 2$.

The information structure, the prices of the securities in periods $t = 0$ and $t = 2$, and their dividends are given by the following event-tree.

Suppose that the optimum consumption plan is to consume \$10 in state ω_2, \$12 in state ω_5 and nothing in other states. We wish to device a trading strategy in the three securities to ønance this plan. How much is the cost of this plan in period zero?

Place youself in event a_1 in period $t = 2$. In order to have \$10 in state ω_2, we

must have a portfolio which contains (x, y, z) units of the three securities so as

$$\begin{aligned} x + 4y + 5z &= 0 \\ x + 5y + 4z &= 10 \\ x + 4y + 4z &= 0 \end{aligned}$$

The solution of these equations gives $x = -40, y = 10, z = 0$. The cost of this portfolio, given the prices in a_1 is \$5.

If event a_2 occurs in $t = 1$, the portfolio of the three securities should give

$$\begin{aligned} x + 6y + 8z &= 0 \\ x + 8y + 6z &= 12 \end{aligned}$$

Here we have more unknowns than equations, so we set, arbitrarily, $z = 0$. The solution to the equations is, then, $x = -36, y = 6, z = 0$. The cost of this portfolio given the prices of the securities in a_2 is \$7.5.

The above means that in $t = 0$ we must have \$5 if a_1 occurs, or \$7.5 if a_2 occurs. To have these amounts we must buy a portfolio in $t = 0$ so as

$$\begin{aligned} x + 4.5y + 4.3z &= 5 \\ x + 7.2y + 6.8z &= 7.5 \end{aligned}$$

Again, we have more unknowns than equations, so we set, arbitrarily, $z = 0$. The solution to the equations is, then, $x = 0.8333, y = 0.9259, z = 0$. The cost of this portfolio given the prices of the securities in $t = 0$ is \$5.9259.

The trading strategy which will implement this consumption plan is
Period $t = 0$: Buy 0.8333 of the ørst security, and 0.9259 units of the second.
Period $t = 1$: If a_1 occurs, short $40 + 0.8333 = 40.8333$ units of the ørst security, and buy 9.0741 $(= 10 - 0.9259)$ units of the second. If a_2 occurs, short $36 + 0.8333 = 36.8333$ units of the ørst security and buy 5.0741 $(= 6 - 0.9259)$ units of the second security.

It should be noted the in the above example there are fewer securities than states. Yet, we were able to implement this particular consumption plan. Following the approach outlined above we can ønd a trading strategy to implement any consumption plan . For that matter any other market security or contigent claim can be duplicated through some trading strategy of these three securities; in other words, the market is complete. What is needed for the completeness of the market is the number of market securities to be at least as many as the number of branches emanating from any branch of the event tree. In that case we say that the market is dynamically complete.

Question: Does the pricing of the third asset admit arbitrage?

9.6 Market Equilibrium

A rational expectations market equilibrium consists I pairs of feasible trading strategies and consumption plans $\{\theta^i, c^i, i = 1, 2, ..., I\}$, and a price process $S(t); t = 0, 1, ..., T\}$ such that θ^i solves the maximization problem of the consumer, c^i is ønanced by θ^i and the markets clear at all times, i.e. if we assume that there is only one unit of each security

$$\sum_{i=1}^{I} \theta_j^i(t) = 1.$$

for all t and all j.

Similarly, the aggregate consumption of a period by all individuals should be equal to total dividends for the period:

$$\sum_{i=1}^{I} c^i(t) = \sum_{j=1}^{N} x_j(t).$$

9.7 Arbitrage Pricing

The description of the market is as in the case of valuation through competitive equilibrium. The pricing equation for the jth security ,which was developed above is repeated here for easy reference.

$$S_j(t) = E\left[\frac{u_{t+1}^{i\prime}(c^i(t+1))}{u_t^{i\prime}(c^i(t))}(x_j(t+1) + S_j(t+1)) \mid F_t\right]$$

or, since $u_t^{i\prime}(c^i(t))$ is known in period t,

$$u_t^{i\prime}(c^i(t))S_j(t) = E\left[u_{t+1}^{i\prime}(c^i(t+1))(x_j(t+1) + S_j(t+1)) \mid F_t\right]$$

Let the $0th$ security be a bond which does not pay dividends until time T, at which time pays one unit of the consumption good in all states, and let $B(t)$ be its price, $t = 0, 1, ..., T$.. $B(t)$ and $S_j(t)$ are random variables measurable with respect to F_t.

The condition for a price system to be an equilibrium one is that it does not permit arbitrage. An arbitrage opportunity is a consumption plan that it is positive at least in one event and has a non-positive initial cost. A price system that admits arbitrage can never be an equilibrium price system because non-satiated individuals will take unbounded positions so that markets cannot clear..

Suppose that in this economy there exists a risk-neutral individual who has no time preference, i.e.

$$u_t^{i\prime}(c^i(t)) = \text{constant for all } s, t.$$

Using this expression in the pricing equation above we obtain:

$$S_j(t) = E\left[x_j(t+1) + S_j(t+1) \mid F_t\right]$$

and

$$B(T) = 1.$$

Deøne the accumulated dividend process for the jth security j to be $D_j(t) = \sum_{s=0}^{t} x_j(s)$, for all $t = 0, 1, 2, ..., T$. Adding this expression to both sides of the previous one we have:

$$S_j(t) + D_j(t) = E\left[S_j(s) + D_j(s) \mid F_t\right]$$

for all $s > t$. Thus, given the palpabilities along the information tree, prices plus accumulated dividends are martingales. **Definition.** A stochastic process $\{X(t), t = 0, 1, , ..., T\}$ is said to be a **martingale** if

$$E[X(t+1)] = X(t), \text{ for } t = 0, 1, ..., T.$$

10 The Pricing of Options

We will use the implication of the no arbitrage condition in order to price options.

An **option** is an agreement which one can purchase and, thus, obtain the right (but not the obligation) to buy or sell a ønancial asset in the future. If the option gives the owner the right to buy a number of shares at a pre-speciøed price, it is refered to as a **call option**; if it gives the right to sell a number of securities, it is a **put option**. The price at which one buys the option is called the **premium**, and the pre-speciøed price at which one can buy or sell the securities is the **exercise price**, or **strike price**. Options can be exrecised during or at the end of a period. If an option can be exercised only at the end of a period, is called a **European Option**; it can be exercised at any time during a period, it is called an **American Option**. Buying an option is said to be holding the option **long**; selling an option which the investor does not own at the time of sale is said to be holding it **short**.

Long options allow the individual to limit his/her losses from an adverse movement in the price of the undelying asset. E.g consider an individual who plans to buy shares in the present with the purpose of selling them at a later time when their

prices increase. If instead he/she buys call options on those shares, he/she can beneøt if the price of shares increase but experience limited losses if the they fall. Similarly, the one who plans to buy shares at some later day (because their price is expected to fall) can hold instead put options which allow the owner to beneøt if indeed the price falls, but experience limitted losses if it increases.

Consider a European call option written at time $t = 0$ on an underlying asset with a market price of S. At time $t = 1$ the buyer of the option has the right to buy the asset for $\$K$ (:the exercise price). The market price of the asset S_T at time $t = 1$ is unknown when the option is written. Obviously, at the expiration date the buyer will exercise the option only if $K < S_T$ because he/she can sell the share in the market and make a proøt $S_T - K$; otherwise he/she will let it expire. Therefore, at expiration the value of the option is

$$f_T \equiv \max[S_T - K, 0]$$

We wish to determine the price (or, premium) which the buyer should by willing to pay for the option today, denoted by f.

10.0.1 Binomial Option Pricing

The Binomial Tree Suppose that there are only two states for $t = 1$, and that the price of the underlying security in state one (S_1) is expected to go up,and in state two (S_2) to go down. In the binomial option pricing it is supposed that the price of the security in each state will be:

$$S_1 = Su \text{ (u for up), } u > 1$$

and

$$S_2 = Sd \text{ (d for down), } d < 1$$

were S is the price of the stock at time zero. (Later in the text we will discuss how u and d are determined in practical applications.) Let the price of the option in each of the two states be $f_1 \equiv f_u$ and $f_2 \equiv f_d$.

The following binomial tree portraits the evolution of the price of the underlying security and the price of the option.

It is called **binomial tree** because of the assumption that there are only two future states which allows the use of binomial formulas. It facilitates the solution of the problem of option pricing, as it will be explained below. For that reason it should be always constructed with the data of the problem, until one becomes proøcient in the pricing of options.

The Pricing of Options-First Method From the no arbitrage theorem, we know that the present :

$$S_0 = \frac{1}{1+r}[\hat{\pi}_1 S_1 + \hat{\pi}_2 S_2]$$

where $\hat{\pi}_s = \phi_s(1+r)$, $s = 1, 2$. Similarly, for the option:

$$f_0 = \frac{1}{1+r}[\hat{\pi}_1 f_u + \hat{\pi}_2 f_d]$$

Thus, to ønd f we must know the prices of the option in each state (f_u and f_d), and the risk-adjusted probabilities ($\hat{\pi}_1$ and $\hat{\pi}_2$) or, equivalently, the state prices (ϕ_1 and ϕ_2).

f_u and $f_d$can be determined from S_1 and S_2 as follows: Suppose that the option will be exercised in the ørst state, but not in the second. Then, $f_u = Su - K$, and $f_d = 0$.

Recall also that state prices can be calculated from the no arbitrage condition and the risk-free interest rate is known, as we have done at the end of t~~he previous~~s 8.2.1 ~~section~~. Therefore, we can determine $\hat{\pi}_1$ and $\hat{\pi}_2$ and we have all the information to determine f_0.

Example In the example of the previous section S_1 and S_2 are given, and there is no arbitrage. This allowed us to calculate the risk-adjusted probabilities $\hat{\pi}_1 = 0.2$ and $\hat{\pi}_2 = 0.8$. Thus, we can estimate the market price of an option with exercise price, say $K = 100$, written on the risky asset of that example:

$$f_1 = \frac{1}{1.1}[0.2 * (150 - 100)] = 9.09.$$

An Alternative Method An alternative method of pricing an option, also based on the no arbitrage condition, is the following.

Consider an option written on the security at time zero which expires at time T 1. Let f be the price (or, premium) of the derivative, f_u its payo□ if u occurs, and f_d if d occurs.

Construct a risk-free portfolio which contains some units of the security and the derivative. Suppose that the portfolio consists of

Δ units of the stock (long).and
-1 unit of the derivative (short)

The cost of the portfolio is:

$$S\Delta - f_0$$

If u occurs, the value of the portfolio is:

$$Su\triangle - f_u$$

If d occurs, its value is:

$$Sd\triangle - f_d$$

The portfolio is risk-free when it yields the same payo□ in both states; i.e. when:

$$S_u\triangle - f_u = S_d\triangle - f_d.$$

Solve this expression for $\triangle$

$$\triangle = \frac{f_u - f_d}{Su - Sd}.$$

This is the number of units that the portfolio must contain in order to be risk-free.

Since the above portfolio is risk-free, it must earn the risk-free rate of return. The present value of the portfolio (with continuous discounting) is:

$$(Su\triangle - f_u)e^{-r\Delta t}$$

Here, r is the annual interest rate and

It is the annual interest rate converted to the rate corresponding to the duration of the option.

$$\Delta t = \frac{1}{N}$$

where N is the number of periods compouding takes place in a year. From the no arbitrage condition, follows that its present value must be equal to its cost, i.e.

$$S\triangle - f = e^{-r\Delta t}[Su\triangle - f_u]$$

or, alternatively:

$$f = e^{-r\Delta t}[\widehat{\pi} f_u + (1 - \widehat{\pi})f_d]$$

$$\text{where } \widehat{\pi} = \frac{e^{r\Delta t} - d}{u - d},$$

and, as before,

$$f_u = Su - K, \text{and } f_d = 0.$$

It can be easily shown that $\widehat{\pi}$ here is the same as $\widehat{\pi}$ in the previous method. For that problem $u = 1.5$, $d = 1$, $r = 0.1$, and (with simple rather than compound discounting)

$$\widehat{\pi} = \frac{1 + r - d}{u - d} = 0.2.$$

Examples: **Problem 1.** A one-step option.

Statement of the problem: $S_0 = 40$, $S_u = 42$, $S_d = 38$, $r = 0.08$ per year, $K = 39$. What is f for a one month European Call Option?

Solution:

Choose the following risk-free portfolio: -1 the option and $+\triangle$ shares of the security. The payo□ of the portfolio must be the same either S_u, or S_d realizes, i.e.

$$42\Delta - 3 = 38\Delta.$$

Solving for Δ, we have $\Delta = 0.75$

The monthly interest rate is $(1/12) * 0.08$,and the value of the portfolio (with monthly compounding) is:

$$28.5 * e^{-0.8*0.0833} = 28.31$$

This must be equal to the cost of the portfolio:

$$-f + \triangle * 40 = 28.31,$$

which gives

$$f = 1.69$$

Alternatively, $\widehat{\pi}$ must satisfy:

$$42\hat{\pi} + 38(1 - \widehat{\pi}) = 40 * e^{0.08*0.0833}$$

i.e.

$$\widehat{\pi} = 0.5669$$

Then:

$$f = e^{-0.08*0.0833}[3 * 0.5669 + 0.4331 * 0]$$

$$f = 1.69.$$

Two-step options In the previous subsections we assumed that the price of the underlying security may go up or down once during a period. Now we will assume that this process may be repeated twice before the option expires. The price of the option at the beginning of time zero is found by applying the one-step method repeatedly, as explained below.

First, construct the binomial tree for both periods, as in the diagram of the example which follows. This gives the evolution of the prices of the security in each period. Place yourself at the end of the second period and derive the price of the option in each terminal branch of the tree from the prices of the security. Then, go to period one and determine the value of the option at each of the two nodes as if each node pertains to a one step option. Finally repeat the previous steps and go to period zero to ønd f.

In other words, ønd ørst f_u and f_d for period one:

$$f_u = e^{-r\Delta t}[\pi f_{uu} + (1-\widehat{\pi}) f_{ud}],$$

and

$$f_d = e^{-r\Delta t}[\pi f_{ud} + (1-\widehat{\pi}) f_{dd}],$$

where r is the annual interest rate and

$$\Delta t = \frac{1}{N}$$

and N is the number of periods that compouding takes place in a year. Then, using these results ønd

$$f_0 = e^{-r\Delta t}[\pi f_u + (1-\widehat{\pi}) f_u]$$

for period zero

The two-steps approach can be combined into one by substituting for f_u and f_d in the last expression. This gives:

$$f = e^{-2r\Delta t}[\widehat{\pi}^2 f_{uu} + 2\hat{\pi}(1-\widehat{\pi}) f_{ud} + (1-\widehat{\pi})^2 f_{dd}].$$

It should be noted that the coe□cients of the $f's$ are the one ontained by expanding the binomial

$$[\widehat{\pi} + (1-\widehat{\pi})]^2.$$

Problem 2. A two step option.

Statement of the problem: The data are as follows: $S = 20$, $u = 1.1$, $d = 0.9$, $r = 0.12$ per anum, $K = 21$. What is the price of a two-step ~~three~~ six month European Call Option? [Note the option is for six months, and the price of the share will be revised twice during that period.]

Note that here $\Delta t = 0.25$ (one quarter). The solution for $\widehat{\pi}$ must satisfy:

$$\widehat{\pi} = \frac{e^{r\Delta t} - d}{u - d} = 0.6523$$

Alternatively, from the price of the share if "up" occurs in period one we deduce that

$$20 * 1.1 = e^{-0.12*0.25}[22 * 1.1\widehat{\pi} + 18 * 0.8(1 - \widehat{\pi})]$$

which solved for $\widehat{\pi}$ also gives: $\widehat{\pi} = 0.6523$.

It is easy to see that

$$f_{uu} = 3.2, \ f_{ud} = 0, \ f_{dd} = 0$$

Then, the value of the option in period one, node B, is:

$$f_u = e^{-0.12*0.25}[0.6523 * 3.2 + 0.3477 * 0] = 2.0257$$

and, in node C:

$$f_d = 0$$

Thus, in period zero, node A:

$$f = e^{-0.12*0.25}[0.6523 * 2.0257 + 0.3477 * 0] = 1.2823$$

Note that here the value of each time step is the same by construction.

The value of u and d in practice Clearly the assumption of only two states of the world is an unrealistic one unless the length of time during which the security is expected to go up or down is a small. For that reason, in practice the time interval is taken to be only a few days or weeks.

Unfortunately, the smaller the interval the larger the number of nodes and branches the binomial tree has, and the more computational demanding is the pricing of options of a given expiration.

To account for small time intervals, u, d, and $\widehat{\pi}$ are chosen as follows.

$$u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}},$$

$$\widehat{\pi} = \frac{e^{r\Delta t} - d}{u - d}$$

where, σ is the stock price volatility (or, variance) and Δt is a very small interval of time.

Multi-Step Payoff of a European Call Option : Suppose on option matures in T periods. That is T is the number of times (or, steps) that the price of the share is revised according to the binomial tree during a year. Use the dynamics of S to go forward along the binomial tree and determine the expiration values of the call option at time T:

$$f_T = \max\ [S_T - K, 0]$$

Then, calculate the the risk-adjusted probabilities as before, and work backwards on the tree for the call option to determine the current value of f .

More speciøcally, at the expiration day the the price of the underlying asset depends on the number of upwards movements of the price of the underlying asset until the option expires. Let that be n, $n = 0, 1, 2, ..., T$, and $S_T = Su^n d^{T-n}$. Then, the payo□ of the option at expiration is:

$$f_T = \max\ [Su^n d^{T-n} - K,\ 0].$$

From the binomial distribution we know that the probability of observing n □up□movements in T periods is

$$C\binom{T}{n}\widehat{\pi}^n(1-\widehat{\pi})^{T-n}$$

where $C\binom{T}{n}$:

$$C\binom{T}{n} \equiv \frac{T!}{n!(T-n)!}$$

is the combination of T things chosen n at a time. In other words, for a given n there are $\frac{T!}{n!(T-n)!}$ ways of observing so many □up□movements in T periods, each one having $\widehat{\pi}^n(1-\widehat{\pi})^{T-n}$ probability to occur. Then, as n varies from zero to T, the persent value of a call at expiration is:

$$f = e^{-r}\{\sum_{n=0}^{T}\frac{T!}{n!(T-n)!}\widehat{\pi}^n(1-\widehat{\pi})^{T-n}\}\max\ [Su^n d^{T-n} - K, 0]\},$$

where:

$$u = e^{\sigma\sqrt{\triangle t}},\ d = e^{-\sigma\sqrt{\triangle t}},\ \text{and}\ \widehat{\pi} = \frac{e^{r\triangle t} - d}{u - d}.$$

Note that if the option expires in a fraction (h) of the year, the appropriae interest rate to discount the value of the $T - step$ option at expiration is hr.

Visually:

Examples Problem 3. Put Options.

Theoretically, the value of a put option at expiration day is

$$g_T = \max[0,\ K - S_T]$$

Statement of the problem: Consider the following problem: $S = 50$, $S_u = 45$, $S_d = 55$, $r = 0.1$, $K = 50$. What is price for a six month put option?

Solution: Consider a risk-free portfolio consisting of Δ units of the underlying asset, and a short sale of one put option.

The cost of the portfolio is

$$50\Delta - g_0$$

The risk-free requirement implies

$$45\triangle - 5 = 55\triangle$$

Solving this for Δ we obtain $\triangle = -0.5$. Thus, the value of the portfolio in six months will be -27.5, and today.

$$-e^{-0.1*0.5} * 27.5 = -26.16.$$

Because of the no arbitrage requirement, the cost must be equal to the value of the portfolio:

$$-g_0 + 50\triangle = -26.16$$

which gives

$$g_0 = 1.16.$$

Alternatively:

$$55\widehat{\pi} + 45(1 - \widehat{\pi}) = 50e^{0.1*0.5}$$

i.e. $\widehat{\pi} = 0.7564$, .and

$$g_0 = [0 * 0.7564 + 5 * 0.2436]e^{-0.1*0.5} = 1.16.$$

Problem 4. One step solution of a two-step call option.

Statement of the problem: $S = 100$, $u = 0.1$, $d = 0.9$, $r = 0.08$, $K = 100$. The up and down movements are expected to take place in each of the two six- month periods. What is f for a European call option.

Solution:

$$\widehat{\pi} = \frac{e^{0.08*0.5} - 0.9}{1.1 - .9} = 0.7041$$

$$e^{-2*0.5*0.08}[0.7041^2 * 21 + 2 * 0.7041 * 0.2959 * 0 + 0.2259^2 * 0] = 9.61$$

10.0.2 Put-Call Parity For European Options

Consider two portfolios:
Portfolio A: One European call option + Cash of value $Ke^{-r(T-t)}$
Portfolio B: One European put option and one share
Both are worth max $[S_T, K]$ at the expiration day. Since they are European options, they cannot be exercised prior to expiration day. Therefore, the portfolios must have identical values today:

$$f_0 + Ke^{-r(T-t)} = g_0 + S_0$$

or,

Call Price + P.V. of Stirike Price = Put Price + Stock Price,

where P.V. stands for present value. This is called the put-call parity. It shows how we can deduce f or g from a put and call option with the same maturity and exercise date.

Problem 5. (Put call parity).

Statement of the problem: Consider the last problem of the previous subsection. What is the value of European put option with $K = 100$?

Solution. Form the values call and put, we have

$$e^{-2*0.5*0.08}[0.7041^2 * 0 + 2 * 0.7041 * 0.2959 * 1 + 0.2259^2 * 19] = 1.92.$$

i.e.

Stock Price + Put Price = 101.92

Present Value of the Strike Price + Call Price = $100e^{-0.08} + 0.961 = 101.92$

10.0.3 Debt and Equity as Options.

Consider a ørm which does not pay any dividends and has a debt which matures T years from now. Let V be the market value of the ørm, S be the value of equity. The debt of the ørm is a zero-coupon risky bond (:the company may go bankrupt)

which has face value D. Shares and bonds may be characterized as call and put options, as explained below.

Shareholders: By issuing bonds it is as if they buy from the bondholders a put option on the ørm and, in addition, they obtain from them a default-free loan. The option has exercise price D, and the value of the loan is equal to the present value of D. If at maturity $V > D$, the put option is out of the money and it is not exercised. The shareholders remain the owners of the ørm and they return the loan (plus interst) to the bondholders. If $V < D$, they exercise the option and sell the shares to the bondholders for $\$D$, which is just enough to cover what they owe to them from the loan.

Bondholders: They have sold a put option on the ørm to the shareholders with exercise price equal to D. In addition, they gave the shareholders a risk-free loan equal to the present value of D. If $V > D$, the the shareholders let the option expire, and the bondholders receive the loan (plus interest). If $V < D$, bondholders have to pay $\$D$ to buy the ørm. This amount is covered from what is owed to them from the loan.

Fron the above it follows that,

$$\text{the value of risky debt} = \text{the value of a risk-free debt } (B) - \text{ the price of a put}(G)$$

From the put-call parity we know that the put of thbe previous paragraph must have a certain relation to a call (F) written on the same asset. To see this consider the following.

Before expiration the value of the ørm is :

$$\begin{aligned} V &= \text{the value of risky debt } + S \\ &= (B - G) + S \end{aligned}$$

from above. On the other hand, the put-call parity says that

$$\begin{aligned} F + De^{-r(T-t)} &= G + S, \text{ or} \\ F + B &= G + S \end{aligned}$$

Combining the two expressions at the time of expiration we obtain:

$$\underbrace{F}_{\text{shares as calls}} = \underbrace{V + G - D}_{\text{shaers as puts}}$$

$$\underbrace{V - F}_{\text{Bonds as calls}} = \underbrace{D - G}_{\text{bonds as puts}}$$

This shows the equivalence of shares and bonds as call or put options.

10.1 American Put & Call Options

10.1.1 Exercise on Calls on a Non-Dividend Paying Stock

Scenario I Suppose that the option expires at time T, today is t: $t < \tau < T$. Consider two portfolios:

A) One American call option + Cash equal to $Ke^{-r(T-t)}$

B) One share

Portfolio A. The value of A at expiration of is equal to the value of cash K, and the value of the option is:

$$\max\left[0, S_T - K_T\right]$$ (value of option only)

Therefore, the value of A if it is held to maturity is:

·either K if the option is not exercised, or,

·$S_T + K_T \; -K_T \; = S_T$ if the option is exercised.

In other words, the value of A at maturity is:

$$\max[S_T, K]$$

If the option is exercised at time $\tau < T$, the value of portfolio A is:

$$\begin{aligned} & S_\tau - K + Ke^{-r(T-\tau)} \\ = \; & S_\tau - K(1 - e^{-r(T-\tau)}) < S \end{aligned}$$

Which is always less than S.

Portfolio B. The value of portfolio B at maturity is S_T, and

$$S_T \ngtr \max[S_T, K].$$

Therefore it does not pay to exercise a call option before T.

If the investor at time τ thinks that the stock is overpriced, then he/she should sell the option rather than exercise it. Before T, **the call option insures the investor against the price falling below K and once it is exercised the insurance is lost.**

It may be possible to exercise a put option on a non-dividend paying stock if it is su□ ciently □deep in the money□.

Scenario II Consider two portfolios:

Portfolio A: One American put option and one share

Portfolio B: Cash of value $Ke^{-r(T-t)}$.

If the put is exercised at $\tau < T$, portfolio A is worth K, while portfolio B is worth $Ke^{-r(T-\gamma)}$. Therefore, A is worth more than B (if exercised).

If the option is held to maturity, A is worth:

$$\max[K, S_T],$$

while B is worth ~~K~~, and which is not clear that it is greater or less than

$$\max\{K, S_T\}.$$

But we cannot argue that an early exercise is not desirable. A put option held in conjunction with the stock insures the owner against the price falling below a certain level. However, unlike the call option, it may optimal for an investor to exercise the put option and forego the insurance in order to realize the strike price immediately.

The early exercise of a put option is more attractive if S decreases, r increases, and σ decreases.

10.2 American Options & The Binomial Tree

Example. Put Option

Statement of the problem: Consider a two year two step put, $K = 52$, $S_0 = 50$. $r = 0.05$, and two time steps: $u = 1.2$ and $d = 0.8$. Find g.

Solution. The value of $\widehat{\pi}$ is:

$$\widehat{\pi} = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05*1} - 0.08}{1.2 - 0.08} = 0.6282$$

Using one equation for both steps:

$$\begin{aligned} g &= e^{-2r\Delta t}[\widehat{\pi}^2 g_{uu} + 2\pi(1-\widehat{\pi})g_{ud} + (1-\widehat{\pi})^2 g_{dd}] \\ &= e^{-2*0.05}[(0.6282^2 * 0) + (2 * 0.6282 * 0.3718 * 4) + (0.3718^2 * 20)] \end{aligned}$$

or, solving for one step at a time using

$$g_u = e^{-r\Delta t}[\pi g_{uu} + (1-\widehat{\pi})g_{ud}].$$

$$g_d = e^{-r\Delta t}[\pi g_{ud} + (1-\widehat{\pi})g_{dd}].$$

and

$$g = e^{-r\Delta t}[\pi g_u + (1-\widehat{\pi})g_d],$$

we derive: $g_{uu} = 0$, $g_{ud} = 4$, $g_{dd} = 20$, $g_u = 1.4147$, $g_d = 0.94636$, $g = 4.1923$.

10.2.1 American Options

The value of the option at the ønal nodes is the same for the American option as it is for the European option. At earlier nodes, the value is:

max [the value given by g_u and g_d, and the payo□ from early exercise]

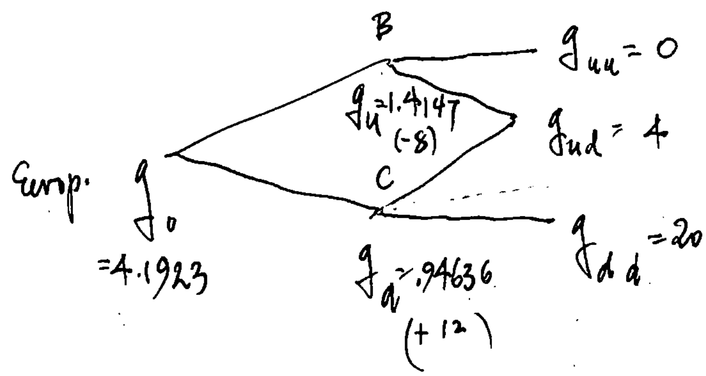
In the above example:

∘ At Point B : $g_u = 1.4147$, and the payo□ from the exercise:$52 - 60 = -8$. Therefor, we do not exercise at point B.

∘ At Point C : $g_d = 0.94636$, and the value from the exercise: $52 - 40 = 12$

∘ At Point A :

$$e^{-0.05*1}[0.6282 * .14147 + 0.3718 * 12] = 5.0894$$



Amer $g_0 = .50894.$